Introduction to Random Matrix Theory II

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- Jean-Marie Normand reviewed elementary properties of RMT
- In addition to the methods he proposed, there are two important methods:
  - the Replica Method
  - The Supersymmetric Method
  - These methods can be used in the case standard methods don't work

- Problems that can be solved: non Gaussian variables, Laplacian Gaussian Matrices, etc...

A simple example: How to compute a resolvant

- Consider a simple problem: Real Random Symmetric Matrix ensemble

- Calculate the resolvant

$$G(z) = \mathrm{Tr}\frac{1}{z-H}$$

where H_{ij} is a $N \times N$ matrix

- Density of states

$$\rho(z) = -\frac{1}{N\pi}\mathrm{Im}G(z+i\epsilon)$$

Gaussian Identities

If A is a symmetric definite positive matrix and u is a real source, one has

$$\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2} x_i A_{ij} x_j + u_i x_i} = \frac{(2\pi)^{N/2}}{(\det A)^{1/2}} e^{\frac{1}{2} u_i A^{-1}_{ij} u_j}$$

which implies

$$A^{-1}_{kl} = \frac{\int \prod_{i=1}^{N} dx_i \; x_k x_l e^{-\frac{1}{2} x_i A_{ij} x_j}}{\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2} x_i A_{ij} x_j}}$$

Resolvant

$$\begin{aligned} G(z) &= \mathrm{Tr}\frac{1}{z-H} \\ &= \sum_i (z-H)^{-1}_{ii} \end{aligned}$$

which can be written as

$$G(z+i\epsilon) = \frac{\int \prod_{i=1}^N dx_i \sum_i x_i^2 e^{\frac{i}{2} x_i (z+i\epsilon-H)_{ij} x_j}}{\int \prod_{i=1}^N dx_i e^{\frac{i}{2} x_i (z+i\epsilon-H)_{ij} x_j}}$$

for real z and positive ϵ

One must in fact compute the average of the resolvant over the random matrix elements

$$G(z) = < \mathrm{Tr} \frac{1}{z - H} >$$

As it is, it is impossible to average over the matrix elements of H

Difficulty comes from the denominator. To escape, two possibilities

Replicas

Supersymmetry

Replicas

In systems with quenched disorder, one must average observables over the disorder distribution

Example: the free energy

$$F = -\frac{1}{\beta} < \log Z >$$

The replica trick (SG: Edwards-Anderson, 1975)

$$< \log Z >= \frac{d}{dn} < Z^n > |_{n=0}$$

Idea: compute for integer *n* and do the analytic continuation at *n=0*

Practically, introduce *n* replicas of the system and perform the average

Replicas

We have

$$G(z+i\epsilon) = \lim_{n\to 0} \int \prod_{i=1}^{N} dx_i \sum_i x_i^2 e^{\frac{i}{2} x_i (z+i\epsilon - H)_{ij} x_j} \left(\int \prod_{i=1}^{N} dx_i \sum_i e^{\frac{i}{2} x_i (z+i\epsilon - H)_{ij} x_j} \right)^{n-1}$$

The limit *n=0* reconstructs the denominator

Idea: compute the quantity for integer *n* and do analytic continuation at *n=0*

$$G(z+i\epsilon) = \lim_{n\to 0} \int \prod_{i=1}^{N} \prod_{\alpha=1}^{n} dx_i^{\alpha} \sum_i x_i^{(1)2} e^{\frac{i}{2} x_i^{\alpha} (z+i\epsilon - H)_{ij} x_j^{\alpha}}$$

$$G(z+i\epsilon) = \lim_{n\to 0} \frac{2}{in} \frac{d}{dz} \int \prod_{i=1}^{N} \prod_{\alpha=1}^{n} dx_i^{\alpha} e^{\frac{i}{2} x_i^{\alpha} (z+i\epsilon - H)_{ij} x_j^{\alpha}}$$

Now it is possible to compute average over the matrices

Supersymmetry

If we use complex variables

$$\int \prod_i \frac{d\phi_i^* d\phi_i}{\pi} e^{-\phi_i^* A_{ij} \phi_j} = (\det A)^{-1}$$

$$\int \prod_i \frac{d\phi_i^* d\phi_i}{\pi} \phi_k^* \phi_l e^{-\phi_i^* A_{ij} \phi_j} = A_{kl}^{-1} (\det A)^{-1}$$

For anticommuting Grasmann variables (Fermionic)

$$\int \prod_i \frac{d\psi_i^* d\psi_i}{\pi} e^{-\psi_i^* A_{ij} \psi_j} = \det A$$

$$\int \prod_i \frac{d\psi_i^* d\psi_i}{\pi} \psi_k^* \psi_l e^{-\psi_i^* A_{ij} \psi_j} = A_{kl}^{-1} \det A$$

Supersymmetry

So the resolvant can be expressed as

$$G(z) = \frac{1}{2} \int \prod_i d\phi_i^* d\phi_i d\psi_i^* d\phi_i \sum_i (\phi_i^* \phi_i + \psi_i^* \psi_i) e^{-\phi_i^* (z-H)_{ij} \phi_j - \psi_i^* (z-H)_{ij} \psi_j}$$

Then the determinants cancel

$$G(z) = -\frac{1}{2} \frac{d}{dz} \int \prod_i d\phi_i^* d\phi_i d\psi_i^* d\psi_i e^{-(z-H)_{ij} (\phi_i^* \phi_j + \psi_i^* \psi_j)}$$

In the following, we will use replicas

Replicas

$$G(z+i\epsilon) = \lim_{n\to 0} \frac{2}{in}\frac{d}{dz}\int \prod_{i=1}^{N}\prod_{\alpha=1}^{n} dx_i^\alpha e^{\frac{i}{2}x_i^\alpha (z+i\epsilon-H)_{ij} x_j^\alpha}$$

Assume the matrix entries are Gaussian identically distributed variables

$$H_{ij} = H_{ji}$$

with

$$P(H_{ii}) = \sqrt{\frac{1}{2\pi H_0^2}} e^{-\frac{H_{ii}^2}{2H_0^2}}$$

$$P(H_{ij}) = \sqrt{\frac{N}{2\pi H^2}} e^{-\frac{NH_{ij}^2}{2H^2}}$$

Note the Wigner scaling which allows good scaling limit

To simplify, assume

$$H_{ii} = 0$$

and perform Gaussian average

$$G(z+i\epsilon) = \lim_{n\to 0} \frac{2}{in}\frac{d}{dz}\int \prod_{i=1}^{N}\prod_{\alpha=1}^{n} dx_i^\alpha e^{\frac{i}{2}(z+i\epsilon)x_i^{(\alpha)2} - \frac{H^2}{2N}\sum_{i<j,\alpha,\beta} x_i^\alpha x_i^\beta x_j^\alpha x_j^\beta}$$

which can be rewritten as

$$G(z+i\epsilon) = \lim_{n\to 0} \frac{2}{in}\frac{d}{dz}\int \prod_{i=1}^{N}\prod_{\alpha=1}^{n} dx_i^\alpha e^{\frac{i}{2}(z+i\epsilon)x_i^{(\alpha)2} - \frac{H^2}{4N}\sum_{\alpha,\beta}(\sum_i x_i^\alpha x_i^\beta)^2}$$

or finally

$$G(z+i\epsilon) = \lim_{n\to 0} \frac{2}{in}\frac{d}{dz}\int \prod_{i=1}^{N}\prod_{\alpha=1}^{n} dx_i^\alpha e^{\frac{i}{2}(z+i\epsilon)x_i^{(\alpha)2} - \frac{H^2}{4N}\sum_{\alpha}(\sum_i x_i^{(\alpha)2})^2 - \frac{H^2}{2N}\sum_{0\le\alpha<\beta\le n}(\sum_i x_i^\alpha x_i^\beta)^2}$$

Using inverse Gaussian transforms,
we get (up to normalizations)

$$e^{-\frac{H^2}{4N}\sum_\alpha(\sum_i x_i^{(\alpha)2})^2} = \int \prod_{\alpha=1}^{n} dq_\alpha e^{-Nq_\alpha^2 - iHq_\alpha \sum_i x_i^{(\alpha)2}}$$

$$e^{-\frac{H^2}{2N}\sum_{0\le\alpha<\beta\le n}(\sum_i x_i^\alpha x_i^\beta)^2} = \int \prod_{\alpha<\beta} dq_{\alpha\beta} e^{-\frac{N}{2}\sum_{\alpha<\beta} q_{\alpha\beta}^2 - iHq_{\alpha\beta}\sum_i x_i^\alpha x_i^\beta}$$

Introduce the $n \times n$ matrix

$$\begin{aligned} Q_{\alpha\alpha} &= 2q_\alpha \\ Q_{\alpha\beta} &= q_{\alpha\beta} \end{aligned}$$

we have

$$G(z+i\epsilon) = \lim_{n\to 0} \frac{2}{in}\frac{d}{dz}\int \prod_{i=1}^{N}\prod_{\alpha=1}^{n} dx_i^\alpha \prod_{\alpha,\beta} dQ_{\alpha\beta} e^{\frac{i}{2}(z+i\epsilon)x_i^{(\alpha)2} - \frac{N}{4}\mathrm{Tr}Q^2 - i\frac{H}{2}Q_{\alpha\beta}\sum_i x_i^\alpha x_i^\beta}$$

Now the replicated variables can be integrated and we obtain

$$G(z+i\epsilon) = \lim_{n\to 0} \frac{2}{in}\frac{d}{dz}\int \prod_{\alpha,\beta} dQ_{\alpha\beta}\; e^{-\frac{N}{2}\left(\frac{1}{2}\mathrm{Tr}Q^2 + \mathrm{Tr}\log(z+i\epsilon - HQ_{\alpha\beta})\right)}$$

where we have used the matrix identity

$$\log\det A = \mathrm{Tr}\log A$$

When $N\to\infty$ we use the saddle-point method

$$Q - H(z-HQ)^{-1} = 0$$

equivalent to the matrix equation

$$HQ^2 - zQ + H = 0$$

This equation determines only the eigenvalues of Q

$$\lambda_{\pm} = \frac{z \pm \sqrt{z^2 - 4H^2}}{2H}$$

This equation does not determine the degeneracy of the eigenvalues.

Let *nq* be the degeneracy of the + solution and *n(1-q)* that of the - solution with *0<q<1*

We have

$$G(z+i\epsilon)=\lim_{n\to 0}\frac{2}{in}\frac{d}{dz}e^{-\frac{nN}{2}\left(\frac{q}{2}\lambda_+^2+\frac{(1-q)}{2}\lambda_-^2+q\log(z+i\epsilon-H\lambda_+)+(1-q)\log(z+i\epsilon-H\lambda_-)\right)}$$

q is chosen so as to maximize the exponent of G

One can see easily that

$$\begin{aligned} q &= 1 \text{ for } z>2H \\ q &= 0 \text{ for } z<-2H \end{aligned}$$

$$G(z+i\epsilon) = 2iN\frac{d}{dz}\left(\frac{q}{2}\lambda_+^2 + \frac{(1-q)}{2}\lambda_-^2 + q\log(z+i\epsilon - H\lambda_+) + (1-q)\log(z+i\epsilon - H\lambda_-)\right)$$

More work (complex analysis) allows to compute the density of states.

One finds of course the semi-circle law

Conclusion

- Simple and elegant method
- Can be generalized to non Gaussian distributions with Wigner scaling
- Allows calculation of corrections to saddle-point
- Can be generalized to many other problems: Optimization, time dependent systems, etc..