Cross talks in the physics of many body systems
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Introduction to random matrix theory

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Problem:

$N \times N$ matrices $\mathbf{M} = (M_{j,k})$ with $M_{j,k}$ random

Statistical properties of eigenvalues (e.v.), eigenvectors ... ?

What happens when $N \to \infty$?

Not yet clear, but success relies on **"universality"**, some law (?) of large number:

"When N large, some properties tend towards statistical properties of a universal limit distribution, depending only on the symmetry properties but not on the details of the probability law of elements"

Outline

1. **Many applications in Physics & Mathematics**
2. **Random matrix (r.m.) model**
3. **Saddle point method**
4. **Orthogonal polynomial method**

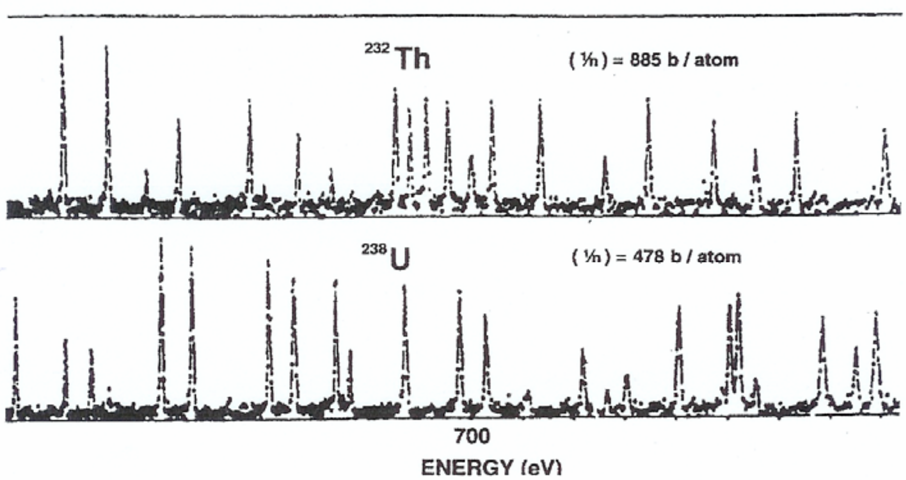
1. Many applications in Physics & Mathematics

Nuclear, Atomic and Molecular Physics

Wigner 1951: energy levels of heavy nuclei, hopeless to describe exactly AND not necessary, cf. Statistical Physics

$H\Psi_n = E_n\Psi_n$ **Hamiltonian** H **not r.**

BUT not well-known AND too complicated for large N



1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 23
d with permission from The American Physical Society, Rahn et al., Neutron re
copy, X, *Phys. Rev. C* 6, 1854–1869 (1972).

One can only study the statistical distribution of levels, observe mean spacing, level density, correlations between levels, ...

Wigner's proposal: the local statistical behavior of levels in a simple sequence (conserved quantities, e.g., J, π) is identical with the eigenvalues of a r.m. **M, what only matter is the symmetry laws.**

Atomic Physics (e.g. H, Li in large B) **Molecular Physics** ...

Symmetries ⇒ 3 ensembles of $N \times N$ r.m. investigated:

- **Gaussian Orthogonal Ensemble, GOE**, $\beta = 1$
time reversal invariant (antiunitary operator) AND J integer OR rotationally invariant

- **Gaussian Unitary Ensemble, GUE**, $\beta = 2$
not time reversal invariant (e.g. with B)

- **Gaussian Symplectic Ensemble, GSE**, $\beta = 4$
time reversal invariant AND J half integer (Kramers degeneracy) AND no rotational symmetry

Repulsion between levels, "Wigner-surmise"

$$\text{Poisson } p(s) \propto e^{-s} \quad \rightarrow \quad p(s) \propto s^{\beta}\, e^{-a\, s^2}$$

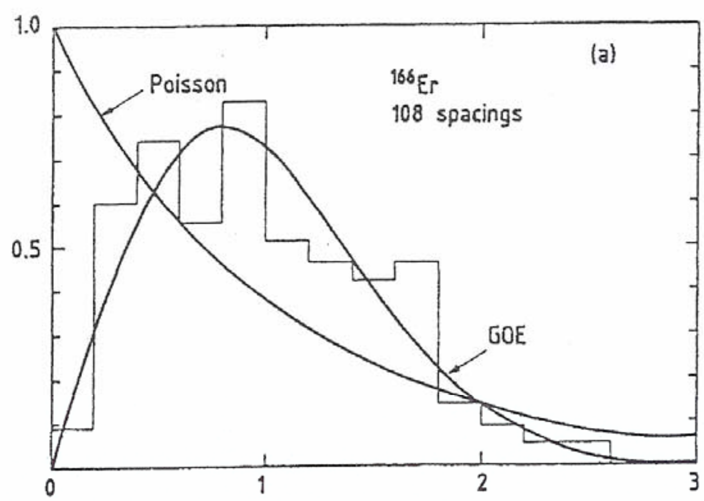


Figure 1.3. The probability density for the nearest neighbor spacings in slow neutron resonance levels of erbium 166 nucleus. The histogram shows the first 108 levels observed. The solid curves correspond to the Poisson distribution, i.e. no correlations at all, and that for the eigenvalues of a real symmetric random matrix taken from the Gaussian orthogonal ensemble (GOE). Reprinted with permission from The American Physical Society, Liou et al., Neutron resonance spectroscopy data, *Phys. Rev. C* 5 (1972) 974–1001.

Quantum Chromodynamics (QCD)

1- $\int$ gluon fluctuation $\sim \langle\ \rangle$, i.e. a particular r.m. theory
$\mathbf{M} = N_c \times N_c$ matrix field of interacting gluons ($N_c = 3$)

Expansion in $1/N_c \Rightarrow$ to each power of $1/N_c$ correspond diagrams with given topology.
$N_c \to \infty$ only planar diagrams ('t Hooft, 1974)

Does not explain confinement BUT **open a new field: random surfaces**

2- Derive some properties of $\mathbf{M}$ = Dirac operator from a simpler model (quadratic) with same symmetries
$\mathbf{M}$ r.m. chiral chGUE, chGOE, chGSE

Discretized surfaces, fields on fluctating surfaces ...

Through **perturbative expansion of random matrices models**

$$Z := \int d\mu(\mathbf{M})\, e^{-N\,\mathrm{Tr}\,V(\mathbf{M})} \qquad V(x) := \frac{g}{2}\,x^2 + \frac{g_3}{3}\,x^3 + \cdots = \frac{g}{2}\,x^2 + \delta V(x)$$

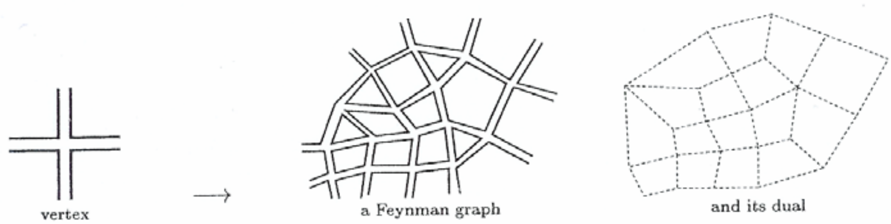
$$= \int d\mu(\mathbf{M})\, e^{-N\,\mathrm{Tr}\,\mathbf{M}^2} \left(1 - N\,\mathrm{Tr}\,\delta V(\mathbf{M}) + \frac{N^2}{2}\left(\mathrm{Tr}\,\delta V(\mathbf{M})\right)^2 + \cdots\right)$$

$$= Z_0 \,\langle\, e^{-N\,\mathrm{Tr}\,\delta V(\mathbf{M})}\,\rangle_0 = 1 - N\,\langle\,\mathrm{Tr}\,\delta V(\mathbf{M})\,\rangle_0 + \frac{N^2}{2}\,\langle\,(\mathrm{Tr}\,\delta V(\mathbf{M}))^2\,\rangle_0 + \cdots$$

Wick's theorem ⇒ Feynmann rules and diagrams

A formal relation: fat graphs (double line propagators) interpreted as a lattice on a surface

Duality diagrams ↔ surfaces



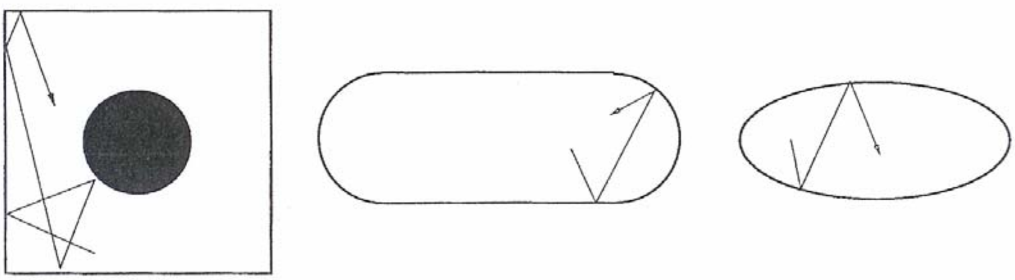
Application in various fields:

- **combinatorics**
- **2D statistical physics systems on random lattices**
- **2D Quantum gravity, string theory**

Quantum chaos

How to characterized a Quantum chaotic system?

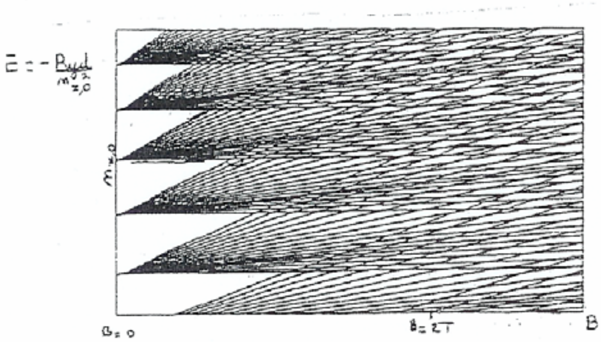
$\mathbf{M} = H$ or evolution operator $\mathbf{U} = e^{-it\mathbf{H}}$, not r. BUT observed and conjecture that when the classical problem is chaotic, the spectrum has the same distribution as a r.m. GUE, GOE, GSE or CUE, COE, CSE.



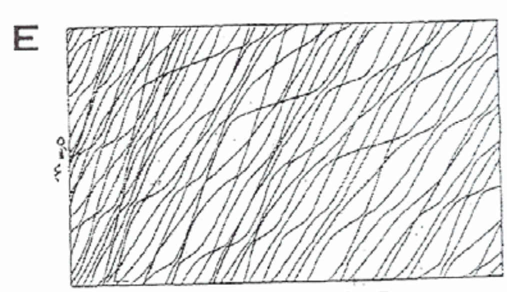
(a) The Sinai's billard (b) The stadium (c) The ellipse

Figure 1.2: (a) The sinai's billiard, and (b) the stadium billiard are both chaotic, while (c) the ellipse is integrable.

Repulsion of e.v.



Champ faible



Champ fort

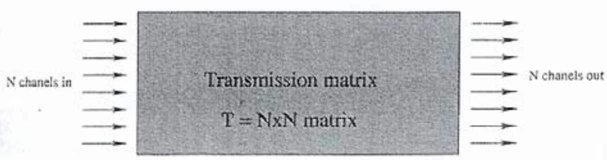
Statistical Physics

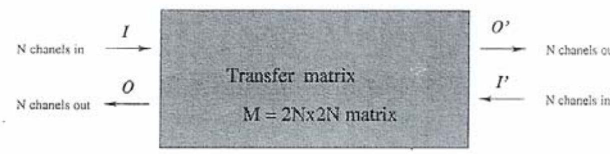
Disordered system

H **naturally r.**, GOE, GUE, GSE, $\langle\,\rangle$ over samples (impurities) or external r. parameter

Transport coef.: electrical, thermal conductance, ...

Mesoscopic physics: e.g. mesoscopic conductors





$$G \propto \mathrm{Tr}\,\mathbf{T}^\dagger\,\mathbf{T} = \mathrm{Tr}\,\frac{2\,\mathbf{M}^\dagger\,\mathbf{M}}{(\mathbf{M}^\dagger\,\mathbf{M}+1)^2}$$

$\mathbf{M}=\mathbf{T}$ $\quad N\times N$ **transmission matrix naturally r.**
$\mathbf{M}=\mathbf{S}$ $\quad 2N\times 2N$ **scattering matrix naturally r.** in $U(N)$
$\mathbf{M}=\mathbf{M}$ $\quad 2N\times 2N$ **transfer matrix naturally r.** in $SU(N,N)$
non-compact group
conservation law, non-compact hyperbolic ensemble TUE, TOE, TSE

$$\mathbf{S} = \begin{pmatrix} \mathbf{R} & \mathbf{T} \\ \mathbf{T}' & \mathbf{R}' \end{pmatrix} \quad \mathbf{S}\begin{pmatrix} I \\ I' \end{pmatrix} := \begin{pmatrix} O \\ O' \end{pmatrix} \quad \mathbf{M}\begin{pmatrix} I \\ O \end{pmatrix} := \begin{pmatrix} O' \\ I' \end{pmatrix}$$

$$|O|^2 + |O'|^2 = |I|^2 + |I'|^2 \qquad |O|^2 - |I|^2 = |I'|^2 - |O'|^2$$

Fields living on a fluctuating r. surfaces

Biological system

e.g. RNA folding where prediction of actual structure from base sequence is challenging
Topological classification of RNA structures (cf. r. surfaces)

Mathematics

Zeros of the Riemann Zeta function

$$\operatorname{Re} z > 1 \quad \zeta(z) := \sum_{n=1}^{\infty} n^{-z} = \prod_{p \text{ prime}} (1 - p^{-z})^{-1}$$

trivial zeros: $z = -2n$, $n = 1, 2, \ldots$

non-trivial zeros in strip $0 < \operatorname{Re} z < 1$, symmetrically / $\operatorname{Re} z = \frac{1}{2}$

Riemann hypothesis (1876): non-trivial zeros on the critical line $\operatorname{Re} z = \frac{1}{2}$
still ?, important in number theory

The distribution of non-trivial zeros $z_n = \frac{1}{2} \pm i\,\gamma_n$, $\gamma_n > 0$ agree with GUE.
Why ?

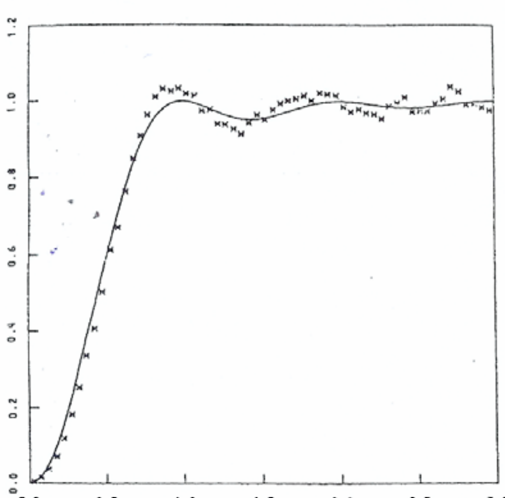


Figure 1.9. Two point correlation function for the zeros $0.5 \pm i\gamma_n$, γ_n real, of the Riemann zeta function; $1 < n < 10^5$. The solid curve is Montgomery's conjecture, Eq. (1.8.9). Reprinted from "On the distribution of spacings between zeros of the zeta function," A.M. Odlyzko. *Mathematics of Computation* pages 273–308 (1987), by permission of The American Mathematical Society.

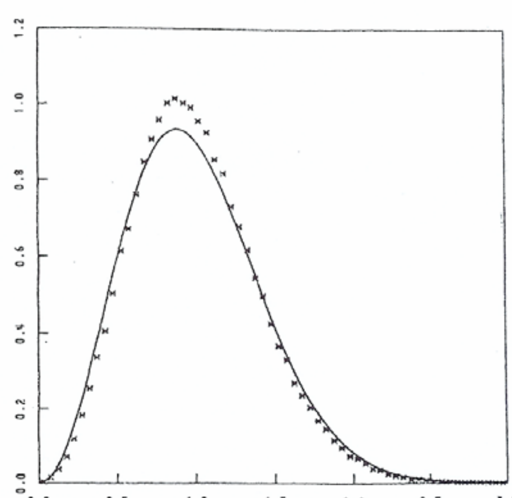


Figure 1.12. Plot of the density of normalized spacings for the zeros $0.5 \pm i\gamma_n$, γ_n real, of the Riemann zeta function on the critical line. $1 < n < 10^5$. The solid curve is the spacing probability density for the Gaussian unitary ensemble, Eq. (6.4.32). From Odlyzko (1987). Reprinted from "On the distribution of spacings between zeros of the zeta function," *Mathematics of Computation* (1987), pages 273–308, by permission of The American Mathematical Society.

Combinatorics, non-commutative probabilities, integrable hierarchies of PDE, ...

2. Random matrix model (e.g. with 1 matrix)

Basic elements

- **A symmetry group** $\mathcal{G}$, e.g.

- **orthogonal group** $\mathrm{O}(N)$, $N \times N$ real orthogonal, $\frac{N(N-1)}{2}$

$$\mathbf{O}\,\mathbf{O}^t = \mathbf{O}^t\,\mathbf{O} = \mathbf{I} \qquad \beta = 1$$

- **unitary group** $\mathrm{U}(N)$, $N \times N$ complex unitary, N^2

$$\mathbf{U}\,\mathbf{U}^\dagger = \mathbf{U}^\dagger\,\mathbf{U} = \mathbf{I} \qquad \beta = 2$$

- **symplectic group** $\mathrm{Sp}(\mathrm{N}, \mathcal{Q}_{\Re})$, $N \times N$ real quaternion symplectic

$$\mathbf{S}\,\overline{\mathbf{S}} = \overline{\mathbf{S}}\,\mathbf{S} = \mathbf{I} \qquad \beta = 4$$

- **An ensemble of $N \times N$ matrices $\mathbf{M} \in \mathcal{E}$, invariant under (an automorphism of) $\mathcal{G}$**

e.g., for $\beta = 1, 2, 4$ respectively

- **real symmetric** $(\mathbf{M} \to \mathbf{O}^t\,\mathbf{M}\,\mathbf{O})$ $\quad \frac{N(N+1)}{2}$

$$M_{j,k} \in \Re,\ \mathbf{M} = \mathbf{M}^t \qquad \mathbf{M} = \mathbf{O}^t\,\Lambda\,\mathbf{O} \qquad \Lambda = \big(x_j\,\delta_{j,k}\big),\ x_j \in \Re$$

- **complex hermitian** $(\mathbf{M} \to \mathbf{U}^\dagger\,\mathbf{M}\,\mathbf{U})$ $\quad N^2$

$$M_{j,k} \in \mathcal{C},\ \mathbf{M} = \mathbf{M}^\dagger \qquad \mathbf{M} = \mathbf{U}^\dagger\,\Lambda\,\mathbf{U} \qquad \Lambda = \big(x_j\,\delta_{j,k}\big),\ x_j \in \Re$$

- **real quaternion self-dual** $(\mathbf{M} \to \overline{\mathbf{S}}\,\mathbf{M}\,\mathbf{S})$ $\quad N(2N-1)$

$$M_{j,k} \in \mathcal{Q}_{\Re},\ \mathbf{M} = \overline{\mathbf{M}} \qquad \mathbf{M} = \overline{\mathbf{S}}\,\Lambda\,\mathbf{S} \qquad \Lambda = \big(x_j\,\delta_{j,k}\big),\ x_j \in \Re$$

Real Quaternions $\Re Q$

4-D associative **non-commutative** algebra over $\Re$

$$q := \sum_{\mu=0}^{3} q^{(\mu)}\, e_\mu = q^{(0)}\, e_0 + \vec{q}.\vec{e} \qquad q^{(\mu)} \in \Re$$

$$e_0\, e_\mu := e_\mu e_0 := e_\mu \qquad \mu = 0,1,2,3$$

$$e_j\, e_k := -\delta_{j,k}\, e_0 + \sum_{\ell=1}^{3} \varepsilon_{j,k,\ell}\, e_\ell \qquad j,k = 1,2,3$$

dual quaternion $\bar{q} := q^{(0)}\, e_0 - \vec{q}.\vec{e}$

$$|q|^2 := q\,\bar{q} = \sum_{\mu=0}^{3} q^{(\mu)2} \geq 0$$

if $q \neq 0 \quad q^{-1} = (q\,\bar{q})^{-1}\,\bar{q}$

$N \times N$ **quaternion** *matrix* $\mathbf{A} := (A_{j,k}) \qquad \mathbf{A} = \sum_{\mu=0}^{3} \mathbf{A}^{(\mu)}\, e_\mu$

$\mathbf{A}^{(\mu)} := \left(A_{j,k}^{(\mu)}\right) \qquad 4N^2$ real parameters

dual matrix $\overline{\mathbf{A}} := (\bar{A}_{k,j}) \qquad \overline{\mathbf{A}\,\mathbf{B}} = \overline{\mathbf{B}}\,\overline{\mathbf{A}}$

self dual $\overline{\mathbf{A}} = \mathbf{A} \quad \Rightarrow \quad \mathbf{A}^{(0)} = \mathbf{A}^{(0)t} \qquad \mathbf{A}^{(j)} = -\mathbf{A}^{(j)t},\ j = 1,2,3$

$N(2N-1)$ real parameters

if $\mathbf{A} = \overline{\mathbf{A}}$ then $\exists \mathbf{S} \in Sp(N, \Re Q)$ symplectic

$$\overline{\mathbf{S}}\,\mathbf{S} = \mathbf{S}\,\overline{\mathbf{S}} = \mathbf{I} \qquad \mathbf{A} = \overline{\mathbf{S}}\,\Lambda\,\mathbf{S} \qquad \Lambda = \left(x_j\,\delta_{j,k}\right),\ x_j \in \Re$$

2 × 2 matrix representation Pauli matrices $\sigma_j\,\sigma_k = \delta_{j,k}\,\mathbf{I} + i\varepsilon_{j,k,\ell}\,\sigma_\ell$

$$\mathbf{c}(e_0) = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{c}(e_1) = i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{c}(e_2) = -i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad \mathbf{c}(e_3) = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$q \rightarrow 2{\times}2 \text{ matrix } \mathbf{c}(q) \quad \text{isomorphism} \quad q_1\,q_2 \rightarrow \mathbf{c}(q_1\,q_2) = \mathbf{c}(q_1)\,\mathbf{c}(q_2)$$

$$\mathbf{c}(\overline{q}) = -\mathbf{z}\,\mathbf{c}(q)^t\,\mathbf{z} \qquad \mathbf{z} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbf{z}^{-1} = -\mathbf{z} = \mathbf{z}^t$$

$$\Rightarrow \quad \mathbf{A}\ N \times N \text{ matrix } \rightarrow \mathbf{C}(\mathbf{A})\ 2N \times 2N \text{ matrix}$$

$$\text{isomorphism} \quad \mathbf{A}_1\,\mathbf{A}_2 \rightarrow \mathbf{C}(\mathbf{A}_1\,\mathbf{A}_2) = \mathbf{C}(\mathbf{A}_1)\,\mathbf{C}(\mathbf{A}_2)$$

$$\mathbf{C}(\overline{\mathbf{A}}) = -\mathbf{Z}\,\mathbf{C}(\mathbf{A})^t\,\mathbf{Z} \qquad \mathbf{Z} := \begin{pmatrix} \mathbf{z} & 0 & \cdots \\ 0 & \mathbf{z} & 0 \\ 0 & 0 & \ddots \end{pmatrix} \quad 2N{\times}2N \text{ block diagonal}$$

$$\mathbf{Z}^{-1} = -\mathbf{Z} = \mathbf{Z}^t$$

$$\mathbf{A} = \overline{\mathbf{A}} \quad \Rightarrow \quad \mathbf{Z}\,C(\mathbf{A}) = -\big(\mathbf{Z}\,C(\mathbf{A})\big)^t \qquad \mathbf{Z}\,C(\mathbf{A}) \text{ antisymmetric}$$

$$N(2N-1) \text{ real parameters}$$

$$\mathbf{S} \in Sp(N, \Re Q) \quad \Rightarrow \quad C(\mathbf{S}) \in Sp(2N, \Re)$$

$$C(\mathbf{S})^t\,\mathbf{Z}\,C(\mathbf{S}) = \mathbf{Z} \qquad \mathbf{Z}\,C(\mathbf{S}) \text{ orthogonal}$$

$$\mathbf{A} = \overline{\mathbf{A}} \quad \Rightarrow \quad C(\mathbf{A}) = C(\overline{\mathbf{S}})\,C(\Lambda)\,C(\mathbf{S})$$

$$C(\Lambda) \text{ real diagonal} \quad x_j \text{ repeated twice}$$

• An invariant measure under $\mathcal{G}$, Haar measure

(Riemannian manifold: metric $\to$ volume): p_γ indep. var.

$$dM = \sum_\gamma \frac{\partial M}{dp_\gamma} dp_\gamma \qquad ds^2(M) = \mathrm{Tr}\big(dM\, dM^\diamond\big) = \sum_{\gamma,\nu} g_{\gamma,\nu}(M)\, dp_\gamma\, dp_\nu$$

$$\Rightarrow \quad d\mu(M) \propto \sqrt{\det g(M)} \prod_\gamma dp_\gamma$$

Ex. p_γ = independent matrix elements $\Rightarrow \quad g(M) = cte$

$$\{M^{(0)}_{j,j},\ 0 \le j \le N-1 \quad M^{(\alpha)}_{j,k},\ 0 \le j < k \le N-1,\ \alpha = 0, \ldots, \beta\}$$

$$d\mu(M) := \prod_j dM^{(0)}_{j,j} \prod_{j<k} \prod_\alpha dM^{(\alpha)}_{j,k} = d\mu(A^\diamond\, M\, A)$$

notations: $\mathbf{A} = \mathbf{O}, \mathbf{U}, \mathbf{S}$ and $\mathbf{A}^\diamond = \mathbf{A}^t, \mathbf{A}^\dagger, \overline{\mathbf{A}}$ for $\beta = 1, 2, 4$

$$\sum_j \equiv \sum_{j=0}^{N-1} \quad \sum_{j<k} \equiv \sum_{0 \le j < k \le N-1} \quad \sum_\alpha \equiv \sum_{\alpha=0}^{\beta} \quad \text{same for } \prod$$

• A probability law $P(\mathbf{M})$ invariant under $\mathcal{G}$

$$\text{invariant } T_n := \frac{1}{N} \operatorname{Tr} \mathbf{M}^n, \; n = 1, \ldots, N$$

$$\Rightarrow \; P(\mathbf{M}) = \mathrm{P}(\{T_n\}) := e^{-N^2 \mathcal{V}(\{T_n\})} \; \sim \; \text{Boltzmann weight}$$

Ex. 1 $V(x) := \frac{g}{2} x^2 + \frac{g_3}{3} x^3 + \cdots \qquad P(\mathbf{M}) := Z^{-1} e^{-N \operatorname{Tr} V(\mathbf{M})}$

One proves, for $\beta = 1, 2, 4$:

$P(\mathbf{M})$ **invariant** AND linearly independent components of $\mathbf{M}$ **statistically independent**

$$P(\mathbf{M}) = P(\mathbf{A}^\diamond \, \mathbf{M} \, \mathbf{A}) \qquad P(\mathbf{M}) = \prod_j f_j^{(0)}(M_{j,j}^{(0)}) \prod_{j<k} \prod_{\alpha=0}^{\beta} f_{j,k}^{(\alpha)}(M_{j,k}^{(\alpha)})$$

$$\Rightarrow \quad P(\mathbf{M}) = e^{-a \operatorname{Tr} M^2 + b \operatorname{Tr} M + c} \quad a > 0, \; b, c \in \Re$$

Gaussian ensembles: $V(x) = \frac{1}{2} x^2$

$\beta = 1$ **Gaussian Orthogonal Ensemble GOE**

$\beta = 2$ **Gaussian Unitary Ensemble GUE**

$\beta = 4$ **Gaussian Symplectic Ensemble GSE**

$$P(\mathbf{M}) := Z^{-1} e^{-N \frac{1}{2} \operatorname{Tr} \mathbf{M}^2} \qquad Z = \int d\mu(\mathbf{M}) \, e^{-N \frac{1}{2} \operatorname{Tr} \mathbf{M}^2} \qquad \int d\mu(\mathbf{M}) \, P(\mathbf{M}) =$$

$$\operatorname{Tr} \mathbf{M}^2 = \sum_{j,k} M_{j,k} M_{k,j} = \sum_{j,k} |M_{j,k}|^2 = \sum_j M_{j,j}^{(0)2} + 2 \sum_{j<k} \sum_{\alpha} M_{j,k}^{(\alpha)2}$$

Ex. 2 Circular ensembles $\mathcal{E} \subset \mathcal{G}$ compact, COE, CUE CSE,

$\beta = 1$ **C. Orthogonal Ensemble COE**, $\mathbf{M} = \mathbf{M}^t \in U(N)$
$\beta = 2$ **C. Unitary Ensemble CUE**, $\mathbf{M} \in U(N)$
$\beta = 4$ **C. Symplectic Ensemble CSE**, $\mathbf{M} = \overline{\mathbf{M}} \in \mathrm{Sp}(N, \mathcal{Q}_{\Re})$

$$\mathbf{M} = \mathbf{A}^{\diamond}\, \Lambda\, \mathbf{A} \quad \lambda_j = e^{i\theta_j},\ \theta_j \in [0, 2\pi[$$

$$P(\mathbf{M}) := Cst. = Z^{-1} \qquad Z = \int d\mu(\mathbf{M}) < \infty$$

Induce law for eigenvalues:
change of variables ⇒ **Jacobian** J, GOE, GUE, GSE

$$\mathbf{M} = \mathbf{A}^{\diamond}\,\Lambda\,\mathbf{A} \qquad \{M_{j,j}^{(0)},\, M_{j<k}^{(\alpha)}\} \to \{\underline{x}_N := \{x_0,\dots,x_{N-1}\},\, d\mu(\mathbf{A})\}$$

$$d\mu(\mathbf{M}) = |J(\underline{x}_N,\mathbf{A})| \left(\prod_j dx_j\right) d\mu(\mathbf{A}) \quad \text{invariance} \;\Rightarrow\; J(\underline{x}_N) \quad \text{study } \mathbf{A} \approx \mathbf{I}$$

$$\mathbf{A}\,\mathbf{A}^{\diamond} = \mathbf{A}^{\diamond}\,\mathbf{A} = \mathbf{I} \quad \Rightarrow \quad d\mathbf{A}\,\mathbf{A}^{\diamond} = -\mathbf{A}\,d\mathbf{A}^{\diamond}$$

$$d\mathbf{M} = \mathbf{A}^{\diamond}\,d\Lambda\,\mathbf{A} + d\mathbf{A}^{\diamond}\,\Lambda\,\mathbf{A} + \mathbf{A}^{\diamond}\,\Lambda\,d\mathbf{A} = \mathbf{A}^{\diamond}\left(d\Lambda + \left[\mathbf{A}\,d\mathbf{A}^{\diamond},\Lambda\right]\right)\mathbf{A}$$

$$d\mathbf{M} \approx d\Lambda + \left[d\mathbf{A}^{\diamond},\Lambda\right] \quad \Rightarrow \quad dM_{j,k}^{(\alpha)} = \delta_{j,k}\,dx_j + (d\mathbf{A}^{\diamond})_{j,k}^{(\alpha)}\,(x_k - x_j)$$

$$J = \begin{vmatrix} \dfrac{\partial M_{j,j}^{(0)}}{\partial x_\ell} = \delta_{j,\ell} & \dfrac{\partial M_{j,k}^{(\alpha)}}{\partial x_\ell} = 0 \\ \dfrac{\partial M_{j,j}^{(0)}}{\partial p_m} = 0 & \dfrac{\partial M_{j,k}^{(\alpha)}}{\partial p_m} \end{vmatrix} = \left(\prod_{j<k}\prod_{\alpha}(x_k - x_j)\right)\det\left[(d\mathbf{A}^{\diamond})_{j,k}^{(\alpha)}\right]$$

$$|J| = \left|\prod_{j<k}(x_k - x_j)\right|^{\beta} d\mu(\mathbf{A}) = |\Delta_N(\underline{x}_N)|^{\beta}\,d\mu(\mathbf{A})$$

$$\Delta_N(\underline{x}_N) := \prod_{j<k}(x_k - x_j) \quad N = 2,3,\dots$$

$$\Rightarrow \quad \int_{\mathcal{E}} d\mu(\mathbf{M})\, P(\mathbf{M})\, F(\Lambda) =$$

$$\left(\int_{\mathcal{G}} d\mu(\mathbf{A})\right) \int_{\mathcal{D}^N} d\underline{x}_N \, |\Delta_N(\underline{x}_N)|^{\beta} \prod_j e^{-N\,V(x_j)}\, f(\underline{x}_N)$$

(Joint) probability density of eigenvalues

$$\mathrm{P}_{N,\beta}(\underline{x}_N) = Z_{N,\beta}^{-1}\, |\Delta_N(\underline{x}_N)|^{\beta} \prod_j e^{-N\,V(x_j)}$$

$$Z_{N,\beta} = \int_{\mathcal{D}^N} d\underline{x}_N \, |\Delta_N(\underline{x}_N)|^{\beta} \prod_j e^{-N\,V(x_j)}$$

$$V(x) := \frac{g}{2}\, x^2 \quad \text{for } GOE,\ GUE,\ GSE$$

Some quantities of interest

- **Partially integrated probability densities**

$$\mathrm{R}_n(\underline{x}_n) := \int_{\mathcal{D}^{N-n}} dx_n \cdots dx_{N-1}\, \mathrm{P}_N(\underline{x}_N) \quad n \le N \qquad \int_{\mathcal{D}^n} d\underline{x}_n\, \mathrm{R}_n(\underline{x}_n) = 1$$

- **n-point correlation functions**

$$\frac{1}{N^n} \langle \prod_{j=0}^{n-1} \mathrm{Tr}\, \delta(x_j - \mathrm{M}) \rangle_N = \frac{1}{N^n} \sum_{j_0,\dots,j_{n-1}} \langle \delta(x_0 - \lambda_{j_0}) \cdots \delta(x_n - \lambda_{j_{n-1}}) \rangle_N$$

$$= \rho_n(\underline{x}_n) + \delta \text{ terms}$$

$$\rho_n(\underline{x}_n) := \frac{1}{N^n} \sum_{\substack{j_0,\cdots j_{n-1} \\ \text{all distinct}}} \langle \delta(x_0 - \lambda_{j_0}) \cdots \delta(x_{n-1} - \lambda_{j_{n-1}}) \rangle_N$$

$$\rho_n(\underline{x}_n) = \frac{N!}{N^n\,(N-n)!} \mathrm{R}_n(\underline{x}_n) \qquad \int_{\mathcal{D}^n} d\underline{x}_n\, \rho_n(\underline{x}_n) = \frac{N!}{N^n\,(N-n)!}$$

Connected n-point correlation functions

Cumulants or cluster functions

$$\rho_{c,n}(\underline{x}_n) := \sum_{\substack{G = G_1 \cdots G_m \\ 1,\dots,n}} (-1)^{n-m}\,(m-1)! \prod_{j=1}^{m} \rho_{G_j}(x_k \in G_j)$$

$$\rho_{c,1}(x_1) = \rho_1(x_1) := \rho(x_1)$$

$$\rho_{c,2}(x_1, x_2) = -\rho_2(x_1, x_2) + \rho(x_1)\,\rho(x_2)$$

$$\rho_{c,3}(x_1, x_2, x_3) = \rho_3(x_1, x_2, x_3) - \Big(\rho(x_1)\,\rho_2(x_2, x_3) + \cdots + \cdots\Big)$$

$$+ 2\,\rho(x_1)\,\rho(x_2)\,\rho(x_3)$$

$$\cdots$$

Level density $n = 1$ (not universal)

$$\rho(x) = \frac{1}{N} \langle \operatorname{Tr} \delta(x - \mathrm{M}) \rangle_N = \frac{1}{N} \sum_j \langle \delta(x - x_j) \rangle_N = \mathrm{R}_1(x)$$

$$\int_{\mathcal{D}} dx\, \rho(x) = 1$$

Local mean spacing between consecutive e.v.

$$\text{in } [x, x + dx] \quad N\rho(x)\, dx \text{ e.v.} \quad \Rightarrow \quad \Delta(x) = \frac{1}{N\, \rho(x)}$$

Counting function and unfolded spectrum

$$D(x) := \frac{1}{N} \langle \sum_j \theta(x - x_j) \rangle_N = \int^x dy\, \rho(y)$$

$$x \to \widetilde{x} := D(x) \qquad \frac{d\widetilde{x}}{dx} = \rho(x) \qquad \widetilde{\rho}(\widetilde{x}) = 1 \qquad \widetilde{\Delta}(\widetilde{x}) = \frac{1}{N}$$

E.g., locally arround 0, $\widetilde{x} \approx x\, \rho(0)$, **to study** $\lim_{N\to\infty}$, $x \to 0$ **at** $\widetilde{x}$ **fixed**

2-point correlation function $n = 2$ ($\rho_{c,2}$ universal)

$$\rho_2(x, y) = \frac{1}{N^2} \sum_{j \neq k} \langle \delta(x - x_j)\, \delta(y - x_k) \rangle_N = \frac{N-1}{N} \mathrm{R}_2(x, y)$$

$$\int_{\mathcal{D}^2} dx\, dy\, \rho_2(x, y) = \frac{N-1}{N}$$

$$\rho_{c,2}(x, y) = -\frac{N-1}{N} \mathrm{R}_2(x, y) + \mathrm{R}_1(x)\, \mathrm{R}_1(y)$$

- **Resolvent or Green function cut along $\mathcal{D}$**

$$W(z) := \frac{1}{N}\langle \operatorname{Tr}\frac{1}{z-\mathbf{M}}\rangle_N = \frac{1}{N}\sum_j \langle \frac{1}{z-x_j}\rangle_N$$

$$= \frac{1}{N}\sum_{n=0}^{\infty}\frac{\langle \operatorname{Tr}\mathbf{M}^n\rangle_N}{z^{n+1}} = \int_{\mathcal{D}} d\lambda\,\rho(x)\,\frac{1}{z-x}$$

$$\frac{1}{N}\langle \operatorname{Tr}\mathbf{M}^n\rangle = \int_{\mathcal{D}} dx\,\rho(x)\,x^n = -\frac{1}{2i\pi}\oint dz\,W(z)\,z^n$$

$$\rho(x) = -\frac{1}{2i\pi}\Big(W(x+i\epsilon) - W(x-i\epsilon)\Big) = -\frac{1}{\pi}\operatorname{Im} W(x+i\epsilon) \qquad \epsilon > 0$$

- **Spacing function**

$E_N(\mathcal{I}, n) :=$ **prob.** $\mathcal{I}$ **contains** n **e.v.** $(0 \le n \le N)$, **e.g.**

$$E_N([x,y],0) := \langle \prod_j \Big(1-\chi(x_j)\Big)\rangle_N \qquad \chi(t) := \begin{cases} 1 & \text{if } t\in[x,y] \\ 0 & \text{otherwise}\end{cases}$$

$$\frac{dE_N([x,y],0)}{dx} = \sum_k \langle \delta(x_k - x)\prod_{j\neq k}\Big(1-\chi(x_j)\Big)\rangle_N$$

$$\frac{d^2E_N([x,y],0)}{dx\,dy} = -\sum_{k\neq\ell}\langle \delta(x_k - x)\,\delta(x_\ell - y)\prod_{j\neq k,\ell}\Big(1-\chi(x_j)\Big)\rangle_N$$

$=$ **prob.** x **and** y **are consecutive e.v.**
$p(s)$ **probability that consecutive levels at distance** s

Main approaches for studying random matrices

(i) Perturbative expansion and discretized surfaces ...
Gaussian $\int$, Wick's theorem, diagrams ...

(ii) saddle point method cf. 3.
- analogy with Coulomb gaz of e.v. in a potential $\sim$ find equilibrium
- gives large N leading contribution to $\int$ in terms of e.v.

(iii) Orthogonal polynomial method cf. 4.
- **exact for finite** N, some results simplify in $N \to \infty$
- analogy with free Fermion gaz in a potential
- available for only some multi-matrix model (open chain $M_j \, M_{j+1}$)

(iv) Replica method cf. latter
- inspire from method for spin glasses $\ln Z = \lim_{n\to\infty} \frac{Z^n - 1}{n}$
- gives large N leading contribution to some matrix integral

(v) Equation of motion
- based on the invariance of an integral under change of variable $\mathbf{M} \to \mathbf{M} + \epsilon \mathbf{M}^\ell$ $\sim$ Ward's identities
- powerful, leads to algebraic equations
- diagrammatic interpretation: "loop equation"

(vi) Supersymetric Efetov's method cf. latter
- only Gaussian potential
- any number of matrices (even ∞, matrix field theory) in any ensemble
- $\langle \det \rangle^n \sim$ with Fermions (anti-commuting Grassman variables) for $n > 0$ and Bosons (commuting variables) for $n < 0$
- efficient for disordered assuming Gaussian potential

(vii) Character expansion or group theory methods
- extension of saddle point method
- decompose the integrand onto a basis of characters, use $\perp$ characters to perform $\int$

(viii) Renormalization method
- $N \to N+1$, integrate last row and column $\Rightarrow$ another $N \times N$ matrix model

3. Saddle point method

Recall for one variable: when $t \to \infty$, dominated by max integrand, i.e. by min $f(x)$

$$t \to \infty \qquad Z(t) := \int_a^b dx\, e^{-t\, f(x)} = e^{-t\, f(x_c)} \sqrt{\frac{2\pi}{t\, f''(x_c)}} \left(1 + \mathrm{O}\left(\frac{1}{t}\right)\right)$$

$$f'(x_c) = 0 \quad f''(x_c) > 0 \quad f(x) = f(x_c) + \frac{1}{2} f''(x_c)\,(x - x_c)^2 + \mathrm{O}\left((x - x_c)^3\right)$$

Naive saddle point does not work

$$N \to \infty \qquad Z_N := \int d\mu(\mathbf{M})\, e^{-N\, \mathrm{Tr}\, V(\mathbf{M})} \approx e^{-N\, \mathrm{Tr}\, V(\mathbf{M}_c)} \;?$$

Problems: zero modes $\mathbf{A}\,\mathbf{M}_c\,\mathbf{A}^{\diamond}$ AND integrand $\propto e^{N^2}$ and N^2 variables

Saddle point for the e.v.
Coulomb gaz analogy $\sim$ Boltzmann weight

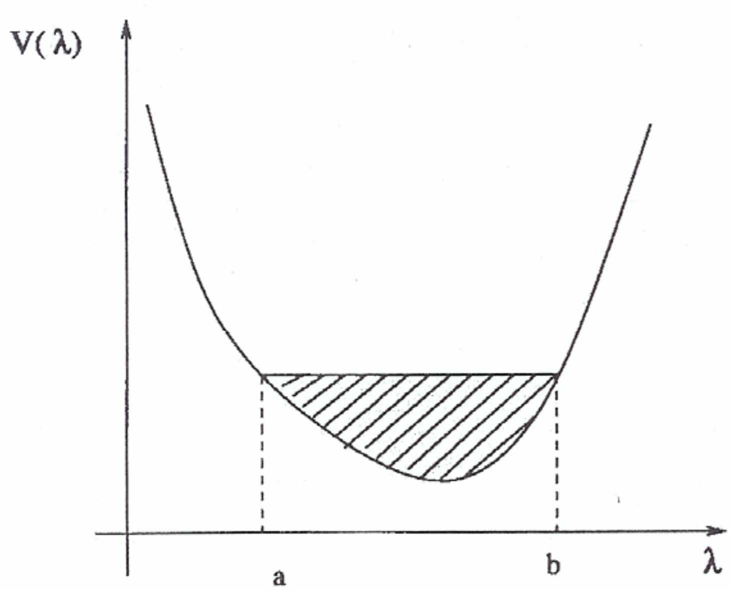
N particles (the e.v.), 1-D space, potential V + **2-D Coulomb interaction resulting from Jacobian** J

$$Z_N := \int_{\mathcal{D}^N} d\underline{x}_N \prod_{j<k} |x_k - x_j|^\beta \, e^{-N\sum_j V(x_j)} = \int_{\mathcal{D}^N} d\underline{x}_N \, e^{-N\,E}$$

$$E := \sum_j V(x_j) - \frac{\beta}{N}\sum_{j<k} \ln|x_k - x_j| \quad F_j = -\frac{\partial E}{\partial x_j} = -V'(x_j) + \frac{\beta}{N}\sum_{k\neq j}\frac{1}{x_j - x_k}$$

Equilibrium state $F_j = 0$, lim $N \to \infty$ for **unique deep well** $\to [a,b]$, continuous limit

$$F_j = 0 \quad \to \quad 0 = -V'(x) + \beta\,\mathrm{PP}\int_a^b dy\,\rho(y)\,\frac{1}{x-y} \quad x \in [a,b] \quad \rho = ?$$



find $\rho(x)$ $\equiv$ find $W(z)$ resolvent
analytical in complex z-plane with a cut $[a,b]$ along $\Re$

$$W(z) := \int_a^b dx\, \rho(y)\, \frac{1}{z-y} \qquad z \to \infty \quad W(z) \sim \frac{1}{z}$$

$$W(x \pm i\epsilon) = \mathrm{PP} \int_a^b dy\, \rho(y)\, \frac{1}{x-y} \mp i\pi\, \rho(x) \quad \Rightarrow$$

$$W(x+i\epsilon) + W(x-i\epsilon) = \frac{2}{\beta}\, V'(x) \quad \text{linear eq.}$$

$$W(z) := \frac{1}{\beta}\, V'(z) + f(z) \quad \Rightarrow \quad f(x+i\epsilon) + f(x-i\epsilon) = 0$$

$$f(z) := -\frac{1}{\beta}\, M(z)\, \sqrt{(z-a)\,(z-b)} \Rightarrow M(x+i\epsilon) = M(x-i\epsilon) \quad \text{analytical}$$

$$W(z) = \frac{1}{\beta}\left(V'(z) - M(z)\, \sqrt{(z-a)\,(z-b)}\right)$$

$$z \to \infty \quad W(z) \sim \frac{1}{z} \qquad \text{with } V(z) \text{ polynomial degree } d$$

$$\Rightarrow M(z) = m_{d-2}\, z^{d-2} + \cdots + m_0 = \text{polynomial part} \frac{V'(z)}{\sqrt{(z-a)\,(z-b)}},\ a,\ b$$

$$\rho(x) = -\frac{1}{2i\pi}\left(W(x+i\epsilon) - W(x-i\epsilon)\right) = \frac{1}{2\pi}\, M(x)\, \sqrt{(x-a)\,(b-x)}$$

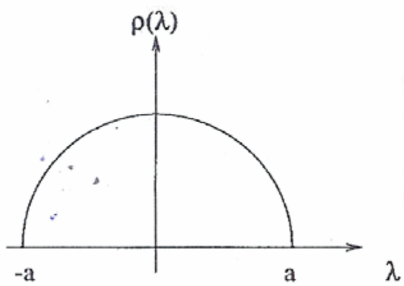
Application to the **Gaussian case:**

$$\mathcal{D} := (-\infty, \infty) \qquad d = 2 \quad V(z) := \frac{1}{2}\, g\, z^2 \qquad V'(z) = g\, z$$

$$z \to \infty \quad W(z) \sim \frac{1}{z} = \frac{1}{\beta}\Big(g\, z - m_0\, z\, (1 - \frac{a+b}{2}\,\frac{1}{z} - \frac{(a-b)^2}{8}\,\frac{1}{z^2} + \mathrm{O}(\frac{1}{z^3}))$$

$$\Rightarrow \quad m_0 = g \qquad a = -b \qquad b^2 = \frac{2\,\beta}{g} \quad \Rightarrow$$

$$\rho(x) = \frac{g}{2\pi}\sqrt{\frac{2\,\beta}{g} - x^2} \quad \textbf{semi} - \textbf{circle law}$$



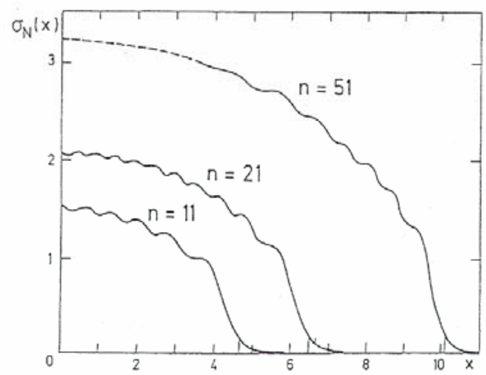


Figure 6.1. The level density $\sigma_N(x)$, Eq. (6.2.10), for $N = 11$, 21 and 51. The oscillations are noticeable even for $N = 51$. The "semi-circle", Eq. (6.2.11), ends at points marked at $\sqrt{22} \approx 4.7$, $\sqrt{42} \approx 6.5$ and $\sqrt{102} \approx 10.1$. Reprinted with permission from E.P. Wigner, "Distribution laws for roots of a random Hermitian matrix" (1962), in *Statistical theories of spectra: fluctuations*, ed. C.E. Porter, Academic Press, New York (1965).

Similar method for $\rho_c(x, y)$ (universal) with 2-point resolvent $W(z, z')$

Trick: introduce temperature T, $V \to V/T$ to get universal quantities

4. Orthogonal polynomial method

GUE Gaussian Unitary Ensemble $\beta = 2$
For simplicity: $\exp(-N\,\frac{g}{2}\,x^2) \to \exp(-x^2)$ with rescalling of x

$$\mathrm{P}_N(\underline{x}_N) = Z_N^{-1}\,\Delta_N(\underline{x}_N)^2 \prod_j w(x_j) \qquad w(x) := e^{-x^2}$$

$$\int_{\mathcal{D}^N} d\underline{x}_N\,\mathrm{P}_N(\underline{x}_N) = 1$$

Trick: "express quantities in terms of determinants"

$$\prod_j w(x_j) = \det\big[w(x_j)\,\delta_{j,k}\big]_N = \det\mathcal{W} \qquad w(x) \geq 0$$

Product of differences = **Vandermonde determinant**

$$\Delta_N(\underline{x}_N) := \prod_{j<k}(x_k - x_j) \;\; N = 2, 3, \ldots$$

$$= \det\big[x_k^j\big]_N = \begin{vmatrix} 1 & 1 & \cdots \\ x_0 & x_1 & \cdots \\ x_0^2 & x_1^2 & \cdots \\ \vdots & \vdots & \vdots \end{vmatrix} = \det\big[\widehat{P}_j(x_k)\big]_N = \Big(\prod_j \frac{\sqrt{h_j}}{k_j}\Big)\,\det\Big[\frac{P_j(x_k)}{\sqrt{h_j}}\Big]_N$$

$\forall$ *monic* $\widehat{P}_j(x) = x^j + \mathrm{O}(x^{j-1})$ or $\forall\ P_j(x) = k_j x^j + \mathrm{O}(x^{j-1})$ $\forall\ h_j > 0$

We will possibly choose P_j orthogonal / measure $\{\mathcal{D}, w\,dx\}$

$$\int_{\mathcal{D}} dx\,w(x)P_j(x)P_k(x) = h_j\,\delta_{j,k}$$

• Expressions of P_N in terms of a determinant

$$\mathrm{P}_N(\underline{x}_N) \propto \det \mathcal{W} \, \det \mathcal{P} \, \det \mathcal{P}$$

$\det(\mathbf{A}\,\mathbf{B}) = \det \mathbf{A} \, \det \mathbf{B}$ and $\det \mathbf{A} = \det \mathbf{A}^t$
$\Rightarrow$ several expressions, not all useful

1- First expression of P_N

Choose $\left(\det\left(\sqrt{\mathcal{W}}\,\mathcal{P}\right)\right)^2$ $\Rightarrow$ in terms of a **Slater determinant, wave function of N Fermions**

$$\varphi_j(x) := \sqrt{w(x)}\,\frac{P_j(x)}{\sqrt{h_j}} \qquad \Phi_N(\underline{x}_N) := \frac{1}{\sqrt{N!}} \det\left[\varphi_j(x_k)\right]_N$$

$$\mathrm{P}_N(\underline{x}_N) = Z_N^{-1} \left(\sqrt{N!} \prod_j \frac{\sqrt{h_j}}{k_j}\right)^2 \left|\Phi_N(\underline{x}_N)\right|^2$$

Choose P_j orthogonal / measure $\{\mathcal{D}, w\,dx\}$ $\Rightarrow$

$$\int_{\mathcal{D}} dx\, \varphi_j(x)\, \varphi_k(x) = \delta_{j,k} \quad \Rightarrow \quad \int_{\mathcal{D}^N} d\underline{x}_N\, \Phi_N(\underline{x}_N)^2 = 1$$

$$\Rightarrow \quad Z_N = N! \prod_j \frac{h_j}{k_j^2} \quad \Rightarrow \quad \mathrm{P}_N(\underline{x}_N) = \Phi_N(\underline{x}_N)^2$$

$\sim$ fundamental N-free Fermion state, Pauli $\Rightarrow$ repulsion
Fermi sea $\varphi_0, \ldots, \varphi_{N-1}$, large N ctrl by Fermi level $\sim N$

2- Second expression of P_N

Choose $\det\left(\sqrt{\mathcal{W}}\,\mathcal{P}^t\,\mathcal{P}\,\sqrt{\mathcal{W}}\right) \Rightarrow$

$$K_N(x,y) := \sqrt{w(x)}\left(\sum_\ell \frac{1}{h_\ell}\,P_\ell(x)\,P_\ell(y)\right)\sqrt{w(y)} = \sum_\ell \varphi_\ell(x)\,\varphi_\ell(y)$$

$$\mathrm{P}_N(\underline{x}_N) = Z_N^{-1}\left(\prod_j \frac{h_j}{k_j^2}\right) det\big[K_N(x_j,x_k)\big]_N$$

$\Rightarrow$ $K_N(x,y)$ **all information about statistic of e.v.**

Orthogonal polynomials allows to compute K_N
Choose P_j **orthogonal / measure** $\{\mathcal{D}, w\,dx\}$ $\Rightarrow$

$$\mathrm{P}_N(\underline{x}_N) = \frac{1}{N!}\,det\big[K_N(x_j,x_k)\big]_N$$

Recurrence relation $P_{n+1} = (A_n x + B_n)P_n - C_n P_{n-1}$
$\Rightarrow$ **Christoffel-Darboux formula**

$$K_N(x,y) = \sqrt{w(x)}\,\frac{k_{N-1}}{k_N\,h_{N-1}}\,\frac{P_N(x)\,P_{N-1}(y) - P_{N-1}(x)\,P_N(y)}{x-y}\,\sqrt{w(y)}$$

$$= \frac{k_{N-1}}{k_N}\sqrt{\frac{h_n}{h_{N-1}}}\,\frac{\varphi_N(x)\,\varphi_{N-1}(y) - \varphi_{N-1}(x)\,\varphi_N(y)}{x-y}$$

$$K_N(x,x) = w(x)\,\sum_\ell \frac{1}{h_\ell}\,P_\ell(x)^2 = \sum_\ell \varphi_\ell(x)^2$$

$$= w(x)\,\frac{k_{N-1}}{k_N\,h_{N-1}}\left(P_{N-1}(x)\,P_N(x)' - P'_{N-1}\,P_N(x)\right)$$

Application to evaluation of correlation functions

Dyson theorem (trick: expand det in permutations decomposed in cycles)

since $\int_{\mathcal{D}} dx\, K_N(x,x) = c$ and $\int_{\mathcal{D}} dy\, K_N(x,y)\, K_N(y,z) = K_N(x,z)$

then $\int_{\mathcal{D}} dx_{N-1} \,\det\big[K_N(x_j,x_k)\big]_N = (c-N+1)\,\det\big[K_N(x_j,x_k)\big]_{N-1}$

i.e. removing the row and the column containing x_{N-1}

Here $c = N \;\Rightarrow$

$$\int_{\mathcal{D}} dx_{N-1}\,\det\big[K_N\big]_N = \det\big[K_N\big]_{N-1}$$

$$\int_{\mathcal{D}} dx_{N-2}\,\det\big[K_N\big]_{N-1} = 2\,\det\big[K_N\big]_{N-2}$$

$$\dots$$

$$\int_{\mathcal{D}} dx_1\,\det\big[K_N\big]_2 = (N-1)\,\det\big[K_N\big]_1 = K_N(x_0,x_0)$$

$$\int_{\mathcal{D}} dx_0\,K_N(x_0,x_0) = N$$

$$\int_{\mathcal{D}} d\underline{x}_N\,\det\big[K_N\big]_N = N! \;\Rightarrow\; \text{recovering} \quad Z_N = N! \prod_j \frac{h_j}{k_j^2}$$

⇒ Partially integrated probability densities or correlation functions (with K_N in terms of ortho. poly.)

$$\mathrm{R}_n(\underline{x}_n) = \int_{\mathcal{D}^{N-n}} dx_n \cdots dx_{N-1}\, \mathrm{P}_N(\underline{x}_n) = \frac{(N-n)!}{N!}\, det\big[K_N(x_j, x_k)\big]_n$$

$$\rho(x) = \mathrm{R}_1(x) = \frac{1}{N}\, K_N(x,x)$$

$$\rho_2(x,y) = \frac{N-1}{N}\, \mathrm{R}_2(x,y) = \frac{1}{N^2}\Big(K_N(x,x)\, K_N(y,y) - K_N(x,y)^2\Big)$$

$$\rho_{c,2}(x,y) = -\rho_2(x,y) + \rho(x)\,\rho(y) = \frac{1}{N^2}\, K_N(x,y)^2$$

More generally

$$\rho_{c,n}(\underline{x}_n) = \frac{1}{N^n} \sum_{\substack{(n-1)! \\ cyclic\ permut.}} K_N(x_{j_0}, x_{j_1})\, K_N(x_{j_1}, x_{j_2}) \cdots K_N(x_{j_{n-1}}, x_{j_0})$$

Gaussian case $\mathcal{D} := (-\infty, \infty) \quad w(x) := e^{-x^2}$

$P_n(x) = H_n(x)$ Hermite $\qquad k_n := 2^n \qquad h_n = 2^n\, n!\, \sqrt{\pi}$

$$H_{n+1} = 2x\, H_n - 2n\, H_{n-1}$$

$$w' = -2x\, w \quad \Rightarrow \quad H_n' = 2n\, H_{n-1} \qquad H_n'' - 2x\, H_n' + 2n\, H_n = 0$$

$$Z_N^{-1} = 2^{-N(N-1)/2} \left(\prod_{j=1}^{N} j!\right) \pi^{N/2}$$

$$\varphi_n(x) = (2^n\, n!\, \sqrt{\pi})^{-\frac{1}{2}}\, e^{-\frac{1}{2}x^2}\, H_n(x)$$

1-D harmonic oscillator orthonormal wave functions

$$h := \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right) \qquad h\,\varphi_n = \epsilon_n\, \varphi_n \quad \epsilon_n = \frac{1}{2} + n$$

$$K_N(x,y) = \frac{1}{2h_{N-1}}\, e^{-\frac{1}{2}(x^2+y^2)}\, \frac{H_N(x)\,H_{N-1}(y) - H_{N-1}(x)\,H_N(y)}{x-y}$$
$$= \sqrt{\frac{N}{2}}\, \frac{\varphi_N(x)\,\varphi_{N-1}(y) - \varphi_{N-1}(x)\,\varphi_N(y)}{x-y}$$
$$K_N(x,x) = \frac{1}{h_{N-1}}\, e^{-x^2}\left(N\,H_{N-1}(x)^2 - (N-1)\,H_{N-2}(x)\,H_N(x)\right)$$
$$= N\,\varphi_N(x)^2 - \sqrt{N(N-1)}\,\varphi_{N-2}(x)\,\varphi_N(x)$$

Known asymptotic behavior of Hermite polynomials ⇒ for $N \to \infty$, see ($g = \frac{2}{N}$)

$$N\,\rho(x) = K_N(x,x) \to \begin{cases} \frac{1}{\pi}\sqrt{2N - x^2} & \text{if } |x| < \sqrt{2N} \\ 0 & \text{otherwise} \end{cases} \quad \textbf{semi-circle law}$$

$$N^2\,\rho_{c,2}(x_1,x_2) = K_N(x_1,x_2)^2$$

with proper scaling $x_j = y_j\,\pi/\sqrt{2N}$ **for** y_j **fixed**

$$\lim_{N\to\infty} \frac{\pi}{\sqrt{2N}}\, K_N(x_1,x_2) = \mathcal{K}(y_1,y_2) = \frac{\sin\pi(y_1-y_2)}{\pi(y_1-y_2)}$$

• Calculation of $\langle$ factorized $F(\Lambda) \rangle$ $\beta = 2$

$$\langle \prod_j f(x_j) \rangle_N := Z_N^{-1} \int_{\mathcal{D}^N} d\underline{x}_N \left|\Delta_N(\underline{x}_N)\right|^2 \prod_j w(x_j)\, f(x_j)$$

$$= Z_N^{-1} \Big(\prod_j \frac{h_j}{k_j^2}\Big)\, N!\ \det\big[F_{j,k}\big]_N$$

$$F_{j,k} := \int_{\mathcal{D}} dx\, f(x)\, \varphi_j(x)\, \varphi_k(x)$$

$$\forall\ P_j(x) := k_j\, x^j + \mathrm{O}(x^{j-1}) \quad \forall\ h_j > 0 \qquad \varphi_j(x) := \sqrt{w(x)}\, \frac{P_j(x)}{\sqrt{h_j}}$$

Proof. Expand both Vandermonde det. and separate variables

$$\Delta_N(\underline{x}_N) \prod_j \sqrt{w(x_j)} = \Big(\prod_j \frac{\sqrt{h_j}}{k_j}\Big) \det\big[\varphi_j(x_k)\big]_N \propto \sum_{\rho \in \mathcal{S}_N} \varepsilon(\rho) \prod_j \varphi_{\rho_j}(x_j)$$

$$\langle\ \rangle_N \propto \sum_{\rho,\sigma \in \mathcal{S}_N} \varepsilon(\rho)\, \varepsilon(\sigma) \prod_{j=0} \int_{\mathcal{D}} dx\, f(x)\, \varphi_{\rho_j}(x)\, \varphi_{\sigma_j}(x)$$

$$\propto \sum_{\rho,\sigma \in \mathcal{S}_N} \varepsilon(\rho)\, \varepsilon(\sigma) \prod_j F_{\rho_j,\sigma_j} = N! \sum_{\rho \in \mathcal{S}_N} \varepsilon(\rho) \prod_j F_{\rho_j,j}$$

$$= Z_N^{-1} \Big(\prod_j \frac{h_j}{k_j^2}\Big)\, N!\ \det[F_{j,k}]_N$$

1- First application: normalization Z_N: $f = 1$ (for $w \geq 0$)

Choose orthogonal polynomials / measure $\{\mathcal{D}, w\, dx\}$

$$\int_{\mathcal{D}} dx\, \varphi_j(x)\, \varphi_k(x) = \delta_{j,k} \Rightarrow\ F_{j,k} = \delta_{j,k}\ \Rightarrow \text{recovering } Z_{N,2} = N! \prod_j \frac{h_j}{k_j^2}$$

2- Second application: about level spacing, calculation of $E_N(\mathcal{I},n)$= prob $\mathcal{I}$ contains n e.v.

$$E_N(\mathcal{I},n) := \binom{N}{n} \langle \Big(\prod_{j=0}^{n-1} \chi_{\mathcal{I}}(x_j)\Big) \Big(\prod_{j=n}^{N-1} (1-\chi_{\mathcal{I}}(x_j))\Big) \rangle_N$$

$$\chi_{\mathcal{I}}(x) := \begin{cases} 1 & \text{if } x \in \mathcal{I} \\ 0 & \text{otherwise} \end{cases}$$

Generating function = $\langle$ factorized $\rangle$

$$R_N(\mathcal{I},z) := \langle \prod_j (1+z\,\chi_{\mathcal{I}}(x_j)\,\rangle_N \qquad E_N(\mathcal{I},n) = \frac{1}{n!}\frac{\partial^n}{\partial z^n} R_N(\mathcal{I},z)\Big|_{z=-1}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\, r_n\, z^n \qquad r_n = \frac{N!}{(N-n)!} \int_{\mathcal{I}^n} d\underline{x}_n\, R_n(\underline{x}_n)$$

$$= Z_N^{-1} \Big(\prod_j \frac{h_j}{k_j^2}\Big)\, N!\ \det[F_{j,k}]_N$$

$$F_{j,k} := \int_{\mathcal{D}} dx\, (1+z\,\chi_{\mathcal{I}}(x))\, \varphi_j(x)\, \varphi_k(x)$$

Choose orthogonal polynomials / measure $\{\mathcal{D}, w\,dx\}$

$$R_N(\mathcal{I},z) = \det\big[F_{j,k}\big]_N = \det\big[\delta_{j,k} + z\,\widetilde{F}_{j,k}\big]_N$$

$$\widetilde{F}_{j,k} := \int_{\mathcal{I}} dx\, \varphi_j(x)\, \varphi_k(x)$$

Compute $\det[F_{j,k}]_N$: consider integral eq. with kernel K_N
$\Rightarrow$Fredholm determinant

$$\int_{\mathcal{I}} dy\, K_N(x,y)\, f(y) = \lambda\, f(x) \qquad K_N(x,y) := \sum_j \varphi_j(x)\, \varphi_j(y)$$

separable kernel $\Rightarrow f(x) = \sum_j f_j\, \varphi_j(x)$ with φ_j linear. indep.

$$\sum_{j,k} f_k\, \varphi_j(x) \int_{\mathcal{I}} dy\, \varphi_j(y)\, \varphi_k(x) = \lambda \sum_j f_j\, \varphi_j(x) \;\Rightarrow\; \sum_k \widetilde{F}_{j,k}\, f_k = \lambda\, f_j$$

$$\big(\widetilde{F}_{j,k}\big) \text{ real sym} \quad \Rightarrow \quad \det\big[\lambda\, \delta_{j,k} - \widetilde{F}_{j,k}\big]_N = \prod_j (\lambda - \widetilde{\lambda}_j)$$

$$\text{setting } \lambda = -\frac{1}{z} \quad \Rightarrow \quad R_N(\mathcal{I}, z) = \prod_j (1 + z\, \widetilde{\lambda}_j)$$

$$n = 0 \qquad E_N(\mathcal{I}, 0) = R_N(\mathcal{I}, z)\big|_{z=-1} = \prod_j (1 - \widetilde{\lambda}_j)$$

Gaussian case:
with proper scaling $x_j = t\, y_j\, \pi/\sqrt{2N}$ for y_j fixed

$$\lim_{N\to\infty} \frac{\pi}{\sqrt{2N}} K_N(x_1, x_2) \;= \mathcal{K}(y_1, y_2) = \frac{\sin \pi\, t\, (y_1 - y_2)}{\pi (y_1 - y_2)}$$

...

1st approx. with 2 levels for GOE, GUE, GSE "**Wigner-surmise**"

$$p(s) \propto \int_{-\infty}^{\infty} dx\, dy\, \delta(|x-y| - s)\, |x-y|^\beta\, e^{-\frac{g}{2}(x^2+y^2)}$$
$$\propto \int_{-\infty}^{\infty} du\, \delta(|u| - s)\, |u|^\beta\, e^{-\frac{g}{4} u^2} \propto s^\beta\, e^{-\frac{g}{4} s^2}$$

• **How to find orthogonal polynomials for given $w(x)$?**
it exists general methods
recurrence equation, integral representation $\langle \det(x - \mathbf{M}) \rangle$, ...

• $\beta = 1, 4$ **Similar methods: problems express** $|\Delta_N(\underline{x}_N)|$ **and** $|\Delta_N(\underline{x}_N)|^4$ **in terms of determinant and/or Pfaffian**

$\Rightarrow$ two **skew-orthogonal scalar products** $(\, , \,)_1$ and $(\, , \,)_4$

$$(f \, , \, g)_2 := \int_{\mathcal{D}} dx \, w(x) \, f(x) \, g(y)$$

$$(f \, , \, g)_1 := \int_{\mathcal{D}^2} dx \, dy \, w(x) \, w(y) \, \mathrm{sgn}(y - x) \, f(x) \, g(y)$$

$$(f \, , \, g)_4 := \int_{\mathcal{D}} dx \, w(x) \Big(f(x) \, g'(y) - f'(x) \, g(x) \Big)$$

• **Generalization to multi-matrix models**, e.g. a chain

$$\mathrm{P}(\mathbf{A}_1, \cdots, \mathbf{A}_n) = Z^{-1} \exp - N \, \mathrm{Tr} \, \mathcal{V}(\mathbf{A}_1, \cdots, \mathbf{A}_n)$$

$$\mathcal{V}(\mathbf{A}_1, \cdots, \mathbf{A}_n) := \sum_{j=1}^{n} V_j(\mathbf{A}_j) - \sum_{j=1}^{n-1} c_j \, \mathbf{A}_j \, \mathbf{A}_{j+1}$$

Conclusion

- Random matrix laws have universality properties
- Random variable laws highly correlated
- They occurs in many phenomena
- Various (beautiful) theoretical methods
- Still many open questions
  A domain of always active researches
- An archetype of fruitful interactions
  between Mathematics and Theoretical Physics

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