# Stochastic Schroedinger Equations


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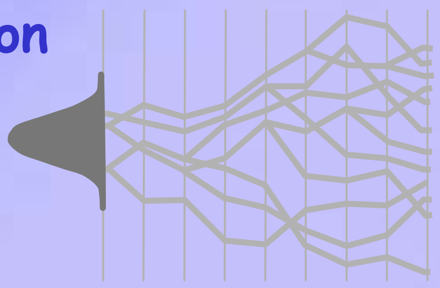
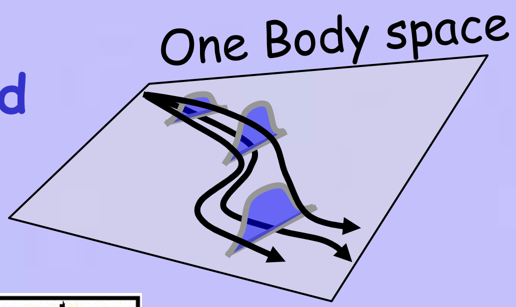
**Part 1: Exact evolution of quantum systems**

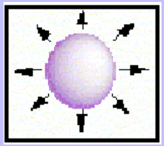
- Introduction to stochastic Schroedinger Equation
- illustration : system-environment
- application to self-interacting system



**Part 2: Approximate evolution of quantum systems**

- Dissipation and fluctuations beyond mean-field
- Quantum jump approach to the many-body problem

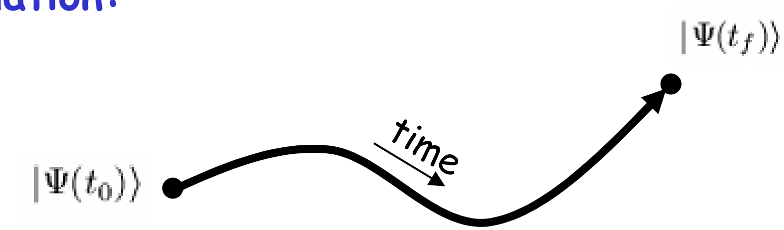


# What is a Stochastic Schroedinger equation?

## Standard Schroedinger equation:

$$d|\Psi\rangle = \frac{dt}{i\hbar} H |\Psi\rangle$$

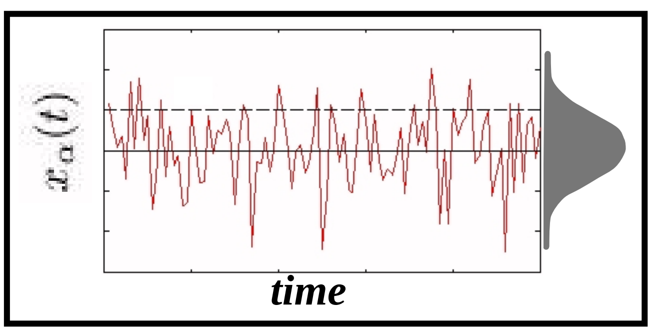


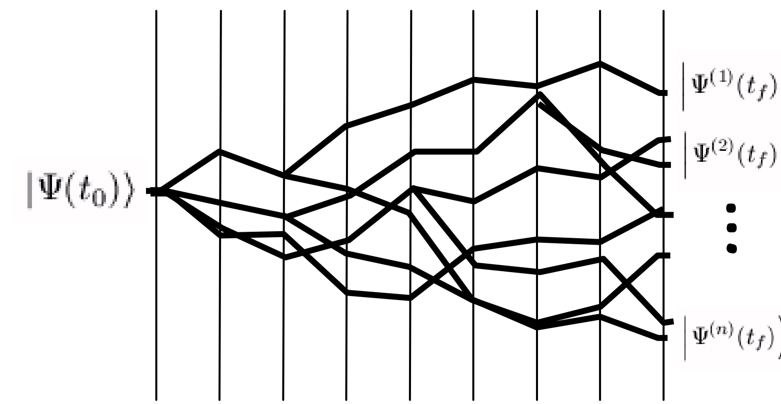

➡ **Deterministic evolution**

## Stochastic Schroedinger equation (SSE):

$$d|\Psi\rangle = \left\{ \frac{dt}{i\hbar} H + dB_{sto} \right\} |\Psi\rangle$$

**Stochastic operator :** $dB_{sto} = \sum_\alpha x_\alpha(t) O_\alpha$

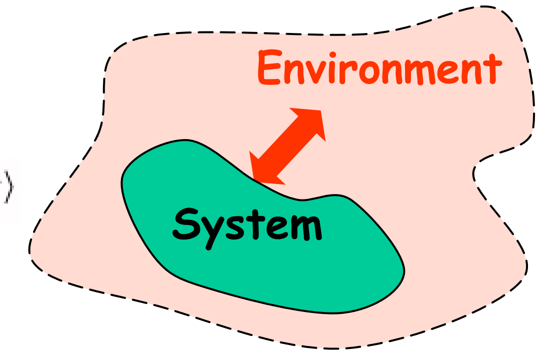


# Exact dynamics of a systems coupled to an environment with SSE

**Hamiltonian**

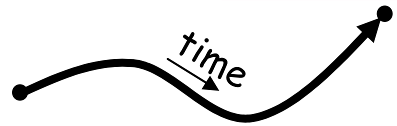
$$H = H_S + H_E + \sum_\alpha B_\alpha(S) \otimes C_\alpha(E)$$




**Exact dynamics**

$$|\Psi(t)\rangle \neq |\Phi\rangle \otimes |\chi\rangle$$

At t=0 $\quad |\Psi(t_0)\rangle = |\Phi(t_0)\rangle \otimes |\chi(t_0)\rangle$




**A stochastic version**

$$| \Psi^{(n)}(t) \rangle = | \Phi^{(n)} \rangle \otimes | \chi^{(n)} \rangle$$

$$\begin{cases} d|\Phi\rangle = \left\{ \frac{dt}{i\hbar} H_S + \sum_\alpha d\xi_\alpha(t) B_\alpha \right\} |\Phi\rangle \\ d|\chi\rangle = \left\{ \frac{dt}{i\hbar} H_E + \sum_\alpha d\xi_\alpha(t) C_\alpha \right\} |\chi\rangle \end{cases}$$

with $\quad \overline{d\xi_\alpha d\xi_\beta} = \frac{dt}{i\hbar}\delta_{\alpha\beta}$


**Average evolution**

$$\overline{d\{|\Phi\rangle \otimes |\chi\rangle\}} = \overline{|d\Phi\rangle \otimes |\chi\rangle} + \overline{|\Phi\rangle \otimes |d\chi\rangle} + \overline{|d\Phi\rangle \otimes |d\chi\rangle}$$

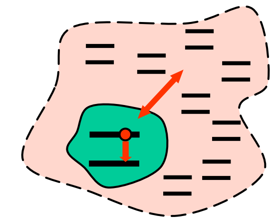
$$\frac{dt}{i\hbar} H_S \quad + \quad \frac{dt}{i\hbar} H_E \quad + \quad \frac{dt}{i\hbar} \sum_\alpha B_\alpha \otimes C_\alpha$$

**The dynamics of the system+environment can be simulated exactly with quantum jumps (or SSE) between "simple" state.**

➡ Average density $\quad D = \overline{|\Psi_1\rangle \langle \Psi_2|}$

# An simple illustration: spin systems

*Lacroix, Phys. Rev. A72, 013805 (2005).*



A two-level system interacting with a bath of spin systems

$$H = 2\sum_{\alpha} C_{\alpha}(\sigma_{+}\sigma_{-}^{(\alpha)} + \sigma_{-}\sigma_{+}^{(\alpha)})$$

system: $\sigma_{+}$, $\sigma_{-}$

environment: $\sigma_{-}^{(\alpha)}$, $\sigma_{+}^{(\alpha)}$

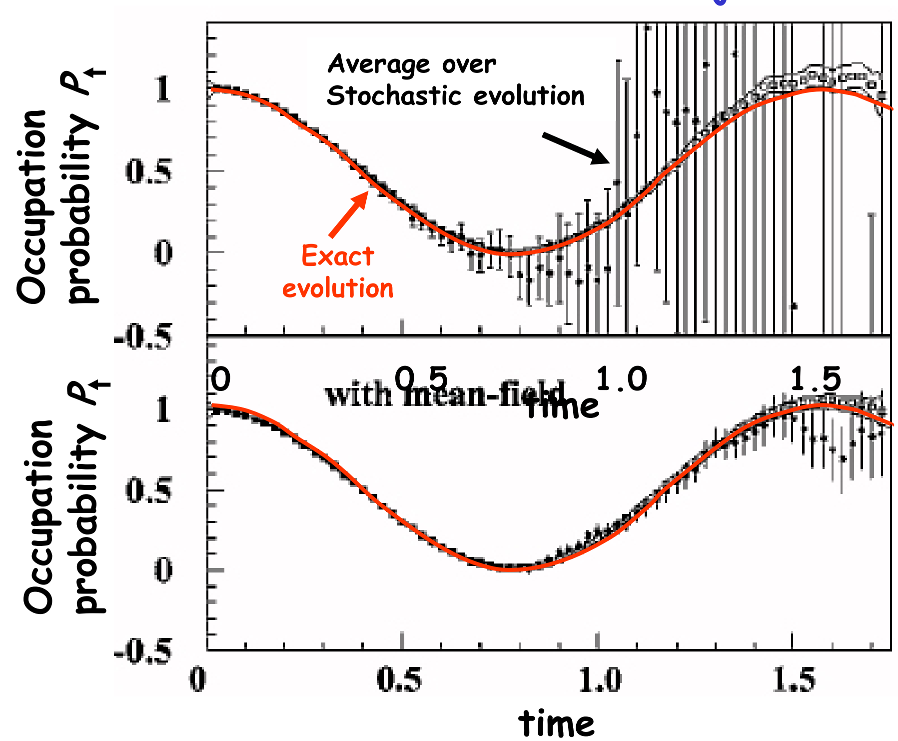
Direct application of SSE:

*H* → "Noise"

Introduction of mean-field:

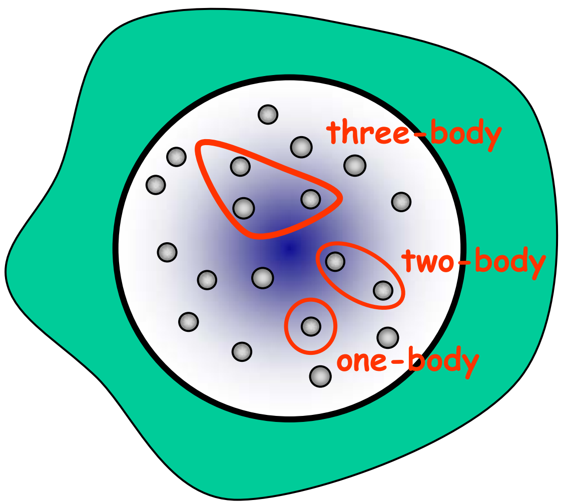
*H* → mean-field + "Noise"

1000 trajectories

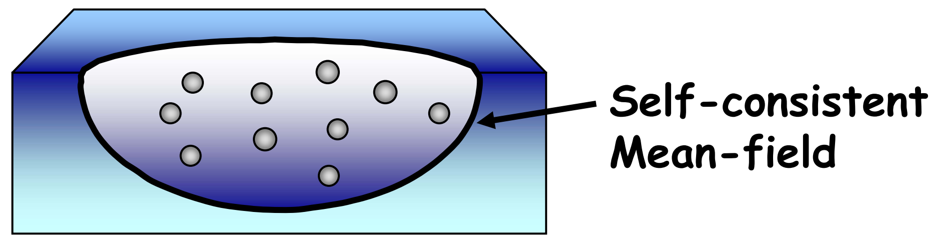



**Stochastic equation are not unique. One can take advantage of this flexibility (mean-field)**

# Simulation of self-interacting system with 'simple state': the nuclei case




**Mean-field: (DFT)**



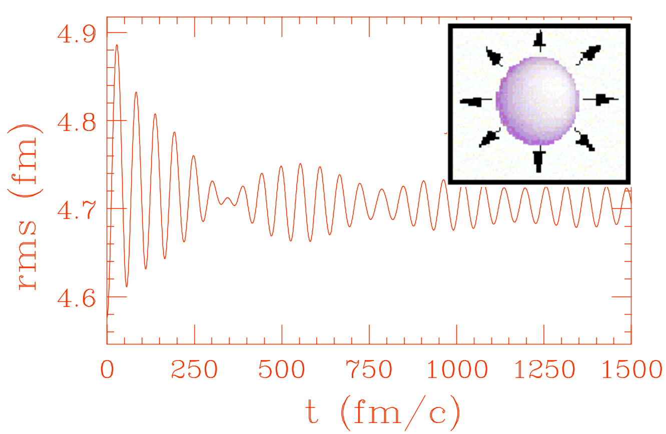

"Simple" Trial state:

$$\Psi(r_1, \cdots, r_{12}, \cdots, r_{123}, \cdots) \longrightarrow |\Phi_{HF}\rangle = \Pi a_\alpha^+ |0\rangle$$

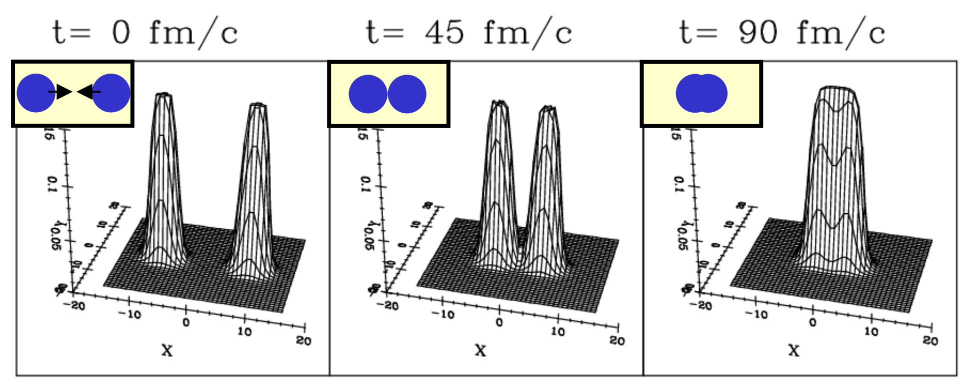
## Success of the mean-field:

## Dynamics

### Vibrations




### Collisions of nucleus




*3D TDHF-Sly4d (P. Bonche)*

**Critical aspects**

- Static: some important long range correlations are neglected.
- Dynamics: correlations (fluctuations) are underestimated.

# Exact Many-Body with SSE on "simple" state: the Functional integral method

## General strategy

*S. Levit, PRC21 (1980) 1594.*

Given a Hamiltonian and an initial State

$$|\Phi(t+\Delta t)\rangle = \exp\left(\frac{\Delta t}{i\hbar}H\right)|\Phi(t)\rangle$$

Write H into a quadratic form

$$H|\Phi\rangle = \left(H_1 - O^2\right)|\Phi\rangle$$

Use the Hubbard Stratonovich transformation

$$\exp\left(-\tfrac{\Delta t}{i\hbar}O^2\right)|\Phi(t)\rangle = \int d\sigma G(\sigma)\exp(a\sigma O)|\Phi(t)\rangle$$

$$\Rightarrow \quad |\Phi(t+\Delta t)\rangle = \int d\sigma G(\sigma)|\Phi_\sigma(t+\Delta t)\rangle$$

Interpretation of the integral in terms of stochastic Schrödinger equation

$$|\Phi_\sigma(t+\Delta t)\rangle = \exp\left(\frac{\Delta t}{i\hbar}H_1 + a\sigma O\right)|\Phi(t)\rangle$$

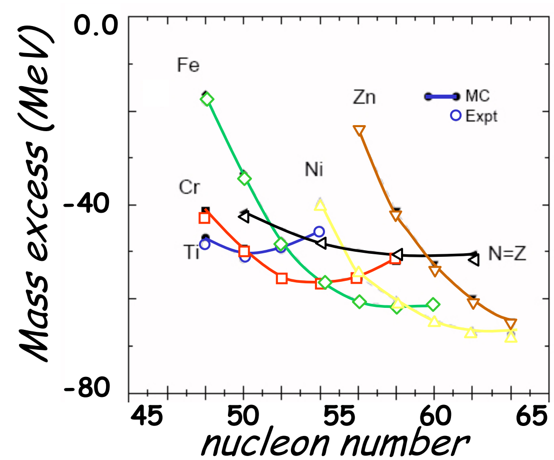
$$\Rightarrow \quad \Delta|\Phi_\sigma\rangle = \left(\frac{\Delta t}{i\hbar}H + a\sigma O\right)|\Phi\rangle$$

The many-body problem

$$H = \sum_{ij} T_{ij} a_i^+ a_j + \frac{1}{4}\sum_{ijkl} V_{ijkl} a_i^+ a_j^+ a_l a_k$$

$a_i^+ a_j \rightarrow O_{ij}$ $\qquad a_i^+ a_j^+ a_l a_k \rightarrow O_{il}\, O_{jk}$

Example of application in nuclear physics:
-Shell Model Monte-Carlo ...




*Adapted from:*
*S.E.Koonin, D.J.Dean, K.Langanke, Ann.Rev.Nucl.Part.Sci. 47, 463 (1997).*

# Recent progress for dynamics: stochastic mean-field

**Functional techniques**

$$|\Phi\rangle = \Pi_\alpha a^+_\alpha |0\rangle$$

$$H = \sum_{ij} T_{ij} a^+_i a_j + \frac{1}{4} \sum_{ijkl} V_{ijkl} a^+_i a^+_j a_l a_k$$

Deterministic part — noise

**New approach: mean-field+ noise**

$$H|\Phi\rangle = \left( \sum_{ij} T_{ij} a^+_i a_j + \frac{1}{4} \sum_{ijkl} V_{ijkl} a^+_i a^+_j a_l a_k \right) |\Phi\rangle$$

Deterministic mean-field — noise Residual inter.

Deterministic evolution

$$D = |\Phi\rangle\langle\Phi|$$

→ Quantum jump between Slater

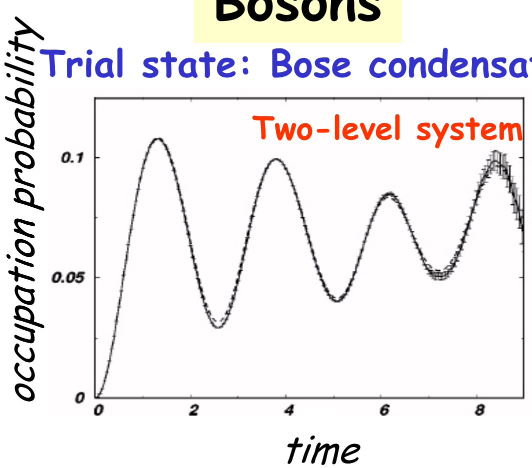
$$D(t) = \overline{|\Phi_a\rangle\langle\Phi_b|}$$

→ Stochastic evolution of single-particle

$$\begin{cases} |d\alpha_a\rangle = (h_{MF}(\rho_a) + dB_a)|\alpha_a\rangle \\ |d\alpha_b\rangle = (h_{MF}(\rho_b) + dB_b)|\alpha_b\rangle \end{cases}$$

## Bosons

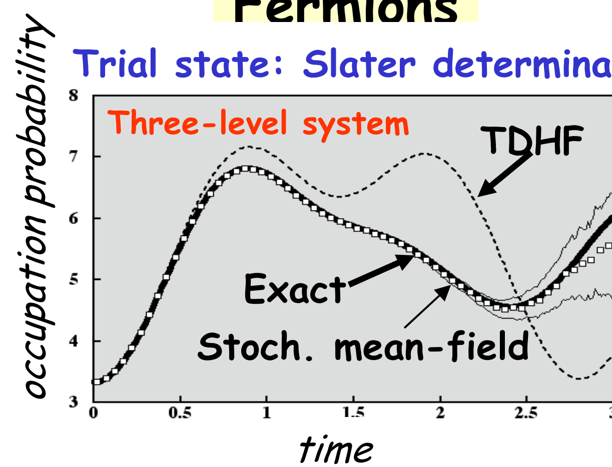
Trial state: Bose condensate




*Carusotto, Y. Castin and J. Dalibard, PRA63 (2001)*

## Fermions

Trial state: Slater determinant




*O. Juillet and Ph. Chomaz, PRL 88 (2002)*

- The link with observable evolution is not simple *(D. Lacroix , PRC71, 064322 (2005))*
- A systematic method is desirable
- The numerical effort is huge

# Mean-field from variational principle

More insight in mean-field dynamics:

**Exact state** $|\Psi(t)\rangle$ → **Trial states**
$$\begin{cases} |Q(t)\rangle \\ |Q+\delta Q\rangle = e^{\sum_\alpha \delta q_\alpha A_\alpha}|Q\rangle \end{cases}$$

**The approximate evolution is obtained by minimizing the action:**

$$S = \int_{t_0}^{t_1} \mathrm{d}s \langle Q | i\hbar\partial_t - H | Q\rangle$$

**Good part: average evolution**

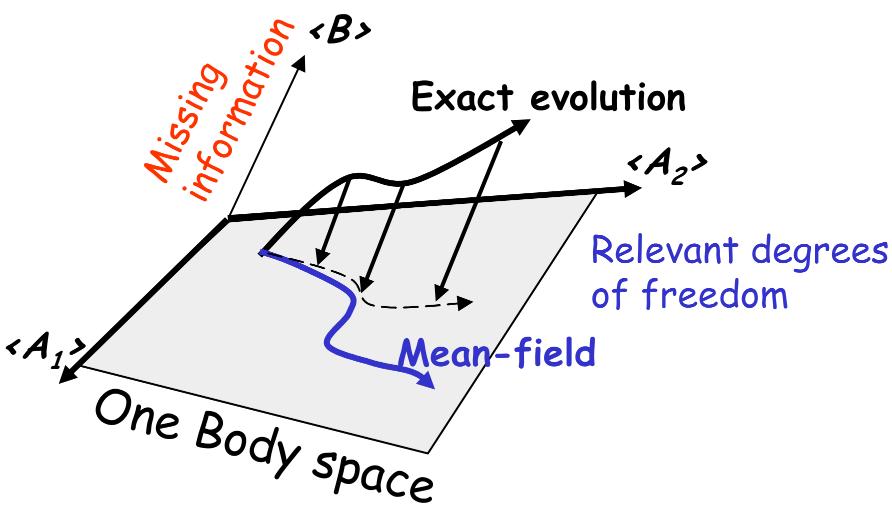
$$i\hbar \frac{\mathrm{d}\langle A_\alpha\rangle}{\mathrm{d}t} = \langle [A_\alpha, H]\rangle$$ → **exact Ehrenfest evolution**

$$H = \mathcal{P}_1 H + (1-\mathcal{P}_1)H$$

**Missing part: correlations**

$$|dQ\rangle = \sum_\alpha dq_\alpha A_\alpha |dQ\rangle = \frac{dt}{i\hbar}\mathcal{P}_1(t) H |Q\rangle$$

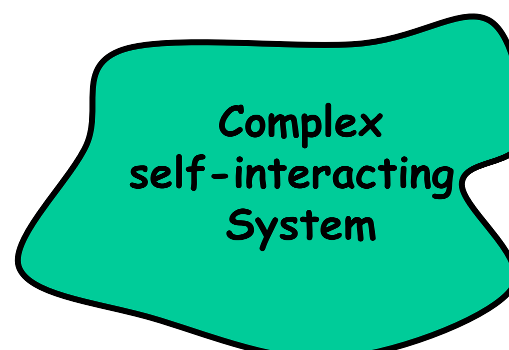
$$\rightarrow \quad i\hbar \frac{d\langle A_\alpha A_\beta\rangle}{dt} \neq \langle [A_\alpha A_\beta, H]\rangle$$




## Hamiltonian splitting

$$H = \mathcal{P}_1 H + (1-\mathcal{P}_1)H$$

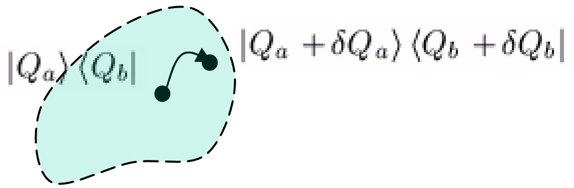
System Environment




The idea is now to treat the missing information as the *Environment* for the Relevant part (*System*)

# Existence theorem : Optimal stochastic path from observable evolution

*D. Lacroix, Annals of Physics (2006), in press.*




**Theorem :**
*One can always find a stochastic process for trial states such that* $\overline{\langle A_\alpha\rangle},\ \overline{\langle A_\alpha A_\beta\rangle},\cdots\overline{\langle A_{\alpha_1}A_{\alpha_2}\cdots A_{\alpha_k}\rangle}$ *evolves exactly over a short time scale.*

**with**

$$\begin{aligned}|Q_a+\delta Q_a\rangle &= e^{\sum_\alpha \delta q_\alpha^{[a]}A_\alpha}\,|Q_a\rangle\\ |Q_b+\delta Q_b\rangle &= e^{\sum_\alpha \delta q_\alpha^{[b]}A_\alpha}\,|Q_b\rangle\end{aligned}$$

*Valid for* $D=|Q_a\rangle\langle Q_b|$

*or* $D=\dfrac{|Q_a\rangle\langle Q_b|}{\langle Q_b\,|\,Q_a\rangle}$

*In practice*

**Mean-field level**

$$\begin{cases}\delta q_\alpha^{[a]}=\delta q_\alpha^{a}\\ \delta q_\alpha^{[b]*}=\delta q_\alpha^{b\,*}\end{cases}$$

$$i\hbar\frac{d}{dt}\langle A_\alpha\rangle = \langle[A_\alpha,H]\rangle$$

**Mean-field + noise**

$$\begin{cases}\delta q_\alpha^{[a]}=\delta q_\alpha^{a}+\delta\xi_\alpha^{[2]}\\ \delta q_\alpha^{[b]*}=\delta q_\alpha^{b\,*}+\delta\eta_\alpha^{[2]}\end{cases}$$

$$i\hbar\frac{d\overline{\langle A_\alpha\rangle}}{dt}=\overline{\langle[A_\alpha,H]\rangle}$$

$$i\hbar\frac{d\overline{\langle A_\alpha A_\beta\rangle}}{dt}=\overline{\langle[A_\alpha A_\beta,H]\rangle}$$

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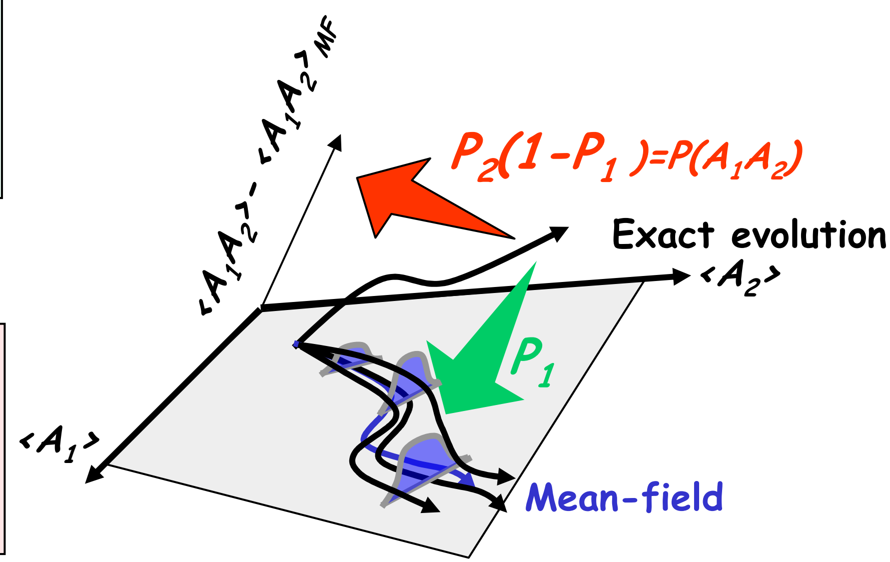
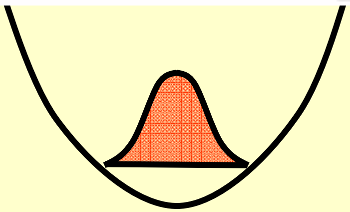


# Simple illustration: simulation of the free wave spreading with "quasi-classical states"

$$H_0 = \hbar\omega(a^+ a + \tfrac{1}{2})$$

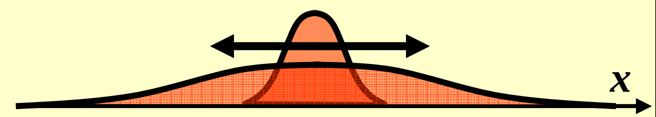
t=0



$$\langle x^2 \rangle = \frac{1}{2\eta} \quad \text{with} \quad \eta = \frac{m\omega}{\hbar}$$

t>0

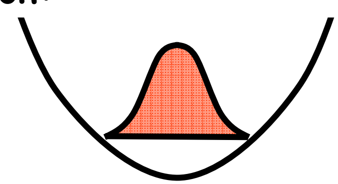
$$H = \frac{p^2}{2m} = -\frac{\hbar\omega}{4}(a^+ - a)^2$$



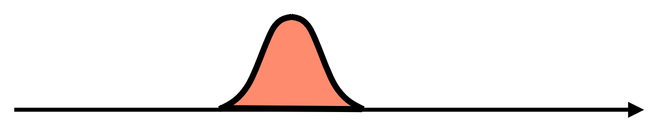

$$\langle x^2 \rangle = \frac{1}{2\eta} + \frac{\hbar^2\eta}{2m^2} t^2$$

**Reduction of the information:** I want to simulate the expansion with Gaussian wave-function having fixed widths. $\langle x^2 \rangle = cte, \quad \langle p^2 \rangle = cte$

Mean-field evolution:



t>0



Relevant/Missing information:

**Relevant degrees of freedom**

$\langle x \rangle, \quad \langle p \rangle$

$\langle a^+ \rangle, \quad \langle a \rangle$

**Missing information**

$\langle x^2 \rangle, \quad \langle p^2 \rangle, \quad \langle xp \rangle$

$\langle a^{+2} \rangle, \quad \langle a^2 \rangle, \quad \langle a^+ a \rangle$

Trial states

$$|Q + \delta Q\rangle = e^{\sum_\alpha \delta q_\alpha A_\alpha} |Q\rangle$$

Coherent states

$$|\alpha + d\alpha\rangle = e^{d\alpha a^+} |\alpha\rangle$$

# Guess of the SSE from the existence theorem

### Densities

$$D = \frac{|\alpha\rangle\langle\beta|}{\langle\beta|\alpha\rangle} \quad \text{with} \quad \begin{array}{l} \langle\beta + \mathrm{d}\beta| = \langle\beta|\mathrm{e}^{\mathrm{d}\beta^* a} \\ |\alpha + \mathrm{d}\alpha\rangle = \mathrm{e}^{\mathrm{d}\alpha a^+}|\alpha\rangle \end{array}$$

### Stochastic c-number evolution from Ehrenfest theorem

$$\begin{cases} \mathrm{d}\alpha = \overline{\mathrm{d}\alpha} + \mathrm{d}\xi^{[2]}, \\ \mathrm{d}\beta^* = \overline{\mathrm{d}\beta^*} + \mathrm{d}\eta^{[2]} \end{cases}$$

mean values

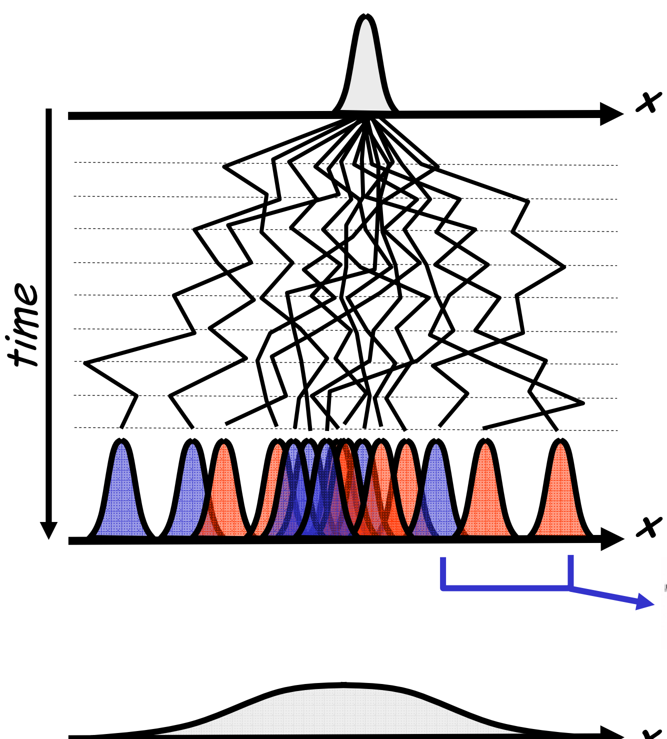
$$\overline{d\langle a\rangle} = \overline{d\alpha}$$

$$\overline{d\langle a^+\rangle} = \overline{d\beta^*}$$

fluctuations

$$\overline{d\langle a^2\rangle} = 2\alpha\overline{d\alpha} + \overline{d\xi^{[2]}d\xi^{[2]}}$$

$$\overline{d\langle a^{+2}\rangle} = 2\beta^*\overline{d\beta^*} + \overline{d\eta^{[2]}d\eta^{[2]}}$$




$$\mathrm{Tr}(Dx^2) = \frac{1}{2\eta} + X^2$$

$$\overline{\mathrm{Tr}(Dx^2)} = \frac{1}{2\eta} + \frac{\hbar^2\eta}{2m^2}t^2$$

### Nature of the stochastic mechanics

$$\begin{cases} X = \frac{1}{\sqrt{2\eta}}(\alpha + \beta^*), \\ P = \mathrm{i}\hbar\sqrt{\frac{\eta}{2}}(\beta^* - \alpha) \end{cases} \Rightarrow \begin{cases} \mathrm{d}X = \frac{P}{m}\,\mathrm{d}t + \mathrm{d}\chi_1 \\ \mathrm{d}P = \mathrm{d}\chi_2, \end{cases}$$

with $\overline{\mathrm{d}\chi_1\,\mathrm{d}\chi_2} = \frac{\hbar^2\eta}{2m}\mathrm{d}t$

**the quantum wave spreading can be simulated by a classical brownian motion in the complex plane**

# SSE for Many-Body Fermions and bosons

*D. Lacroix, Annals of Physics (2006), in press.*

**Starting point:**

$$H = \sum_{ij} \langle i|T|j\rangle a_i^+ a_j + \frac{1}{2}\sum_{ijkl} \langle ij|v_{12}|lk\rangle a_i^+ a_j^+ a_l a_k$$

$$D_{ab} = |\Phi_a\rangle\langle\Phi_b|$$ **with** $$\langle\Phi_b \mid \Phi_a\rangle = 1$$

$$\rho_1 = \sum |\alpha_i\rangle\langle\beta_i|$$

**Observables** $\langle j|\rho_1|i\rangle = \langle a_i^+ a_j\rangle$

**Fluctuations** $\langle ij|\rho_{12}|kl\rangle = \langle a_k^+ a_l^+ a_j a_i\rangle$

**Ehrenfest theorem ➡ BBGKY hierarchy**

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\rho_1 = [h_{\mathrm{MF}}, \rho_1],$$

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\rho_{12} = [h_{\mathrm{MF}}(1) + h_{\mathrm{MF}}(2), \rho_{12}] + (1-\rho_1)(1-\rho_2)v_{12}\rho_1\rho_2 - \rho_1\rho_2 v_{12}(1-\rho_1)(1-\rho_2)$$

$$v_{12} = \sum_\lambda O_\lambda(1)O_\lambda(2)$$

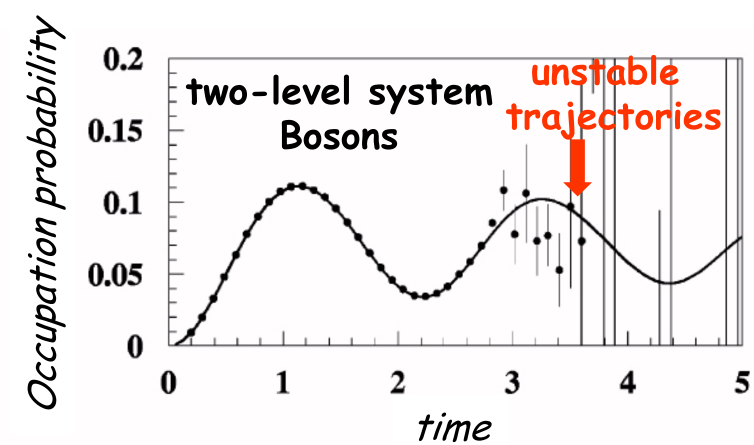
**Stochastic one-body evolution**

$$d\rho_1 = [h_{MF}, \rho_1] + \sum_\lambda d\xi_\lambda^{[2]}(1-\rho_1)O_\lambda\rho_1 + \sum_\lambda d\eta_\lambda^{[2]}(1-\rho_1)O_\lambda\rho_1$$

**with** $$\overline{d\xi_\lambda^{[2]} d\xi_{\lambda'}^{[2]}} = -\overline{d\eta_\lambda^{[2]} d\eta_{\lambda'}^{[2]}} = \delta_{\lambda\lambda'}\frac{dt}{i\hbar}$$

- The method is general.
  the SSE are deduced easily
  ➡ extension to Stochastic TDHFB
  *D. Lacroix, arXiv nucl-th 0605033*
- The mean-field appears naturally and the interpretation is easier
- the numerical effort can be reduced by reducing the number of observables

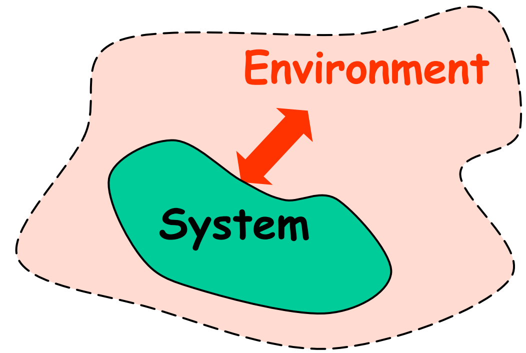
but...



# Part II
# Dissipation in Many-Body Systems with SSE

# Quantum jump method -Dissipation

$$H = H_S + H_E + \sum_\alpha B_\alpha(S) \otimes C_\alpha(E)$$




## Exact dynamics

with SSE on simple state $|\Psi\rangle = |\Phi\rangle \otimes |\chi\rangle$

$$|\Psi^{(n)}(t)\rangle = |\Phi^{(n)}\rangle \otimes |\chi^{(n)}\rangle$$


Then, the average dyn. identifies with the exact one

① For total wave $\overline{d|\Psi\rangle} = \left\{ \frac{dt}{i\hbar} H + \mathcal{O}(dt) \right\} |\Psi\rangle$

② For total density $D = \overline{|\Psi_1\rangle \langle \Psi_2|}$

Application to self-interacting system
Interpretation as a "system+environment"

## Approximate Dissipative dynamics

At t=0 $D(t=0) = \rho_S \otimes \rho_E$

- Weak coupling approx.
- Projection technique
- Markovian approx.

Lindblad master equation:

$$i\hbar \frac{d}{dt}\rho_S = [H_S, \rho_S] + \sum_k \gamma_k \left( A_k A_k \rho_S + \rho_S A_k A_k - 2 A_k \rho_S A_k \right)$$

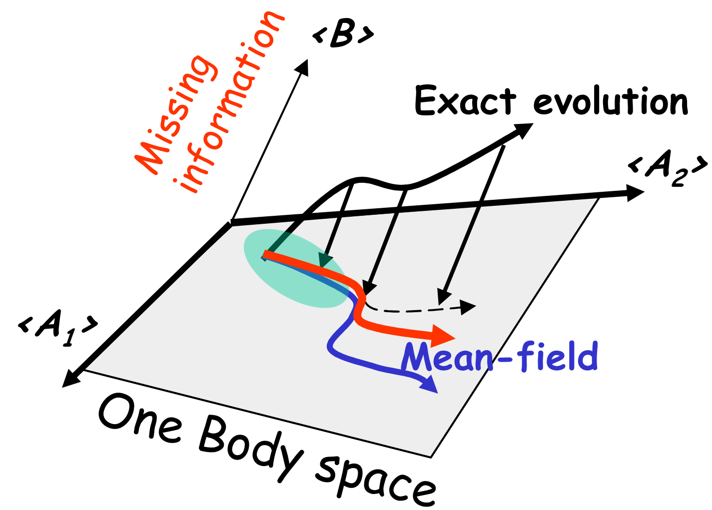
Can be simulated by stochastic eq. on $|\Phi>$,
The Master equation being recovered using :

$$\rho_S = \overline{|\Phi\rangle \langle \Phi|}$$

Gardiner and Zoller, *Quantum noise* (2000)
Breuer and Petruccione, *The Theory of Open Quant. Syst.*

# Dissipation in self-interacting systems

*Y. Abe et al, Phys. Rep. 275 (1996)*
*D. Lacroix et al, Progress in Part. and Nucl. Phys. 52 (2004)*




## *Short time evolution*

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\rho_1 = [h_{\mathrm{MF}}, \rho_1],$$

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\rho_{12} = [h_{\mathrm{MF}}(1) + h_{\mathrm{MF}}(2), \rho_{12}]$$
$$+ (1-\rho_1)(1-\rho_2)v_{12}\rho_1\rho_2 - \rho_1\rho_2 v_{12}(1-\rho_1)(1-\rho_2)$$

**Correlation**

$$C_{12} = \rho_{12} - (\rho_1\rho_2)_A$$

## *Approximate long time evolution+Projection*

$$i\hbar\frac{d}{dt}\rho_1 \;=\; [h_{MF}, \rho_1] + Tr_2\,[v_{12}, C_{12}]$$

**with**

$$C_{12}(t) = -\frac{i}{\hbar}\int_{t_0}^{t} U_{12}(t,s)\,F_{12}(s)\,U_{12}^{\dagger}(t,s)\,ds + \delta C_{12}(t)$$

projected two-body effect

Propagated initial correlation

**Dissipation**

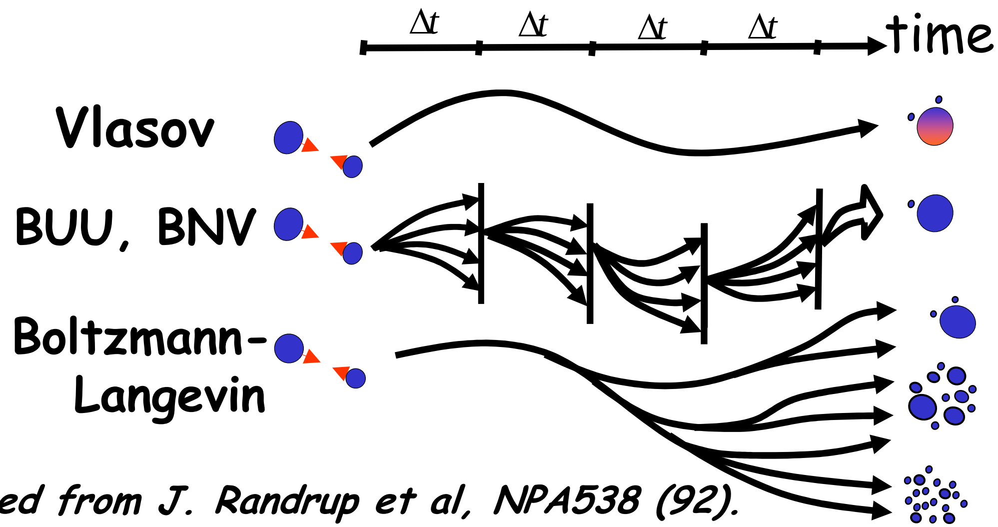
$$i\hbar\frac{d}{dt}\rho \;=\; [h_{MF}, \rho] + K(\rho)$$

**Dissipation and fluctuation**

$$i\hbar\frac{d}{dt}\rho \;=\; [h_{MF}, \rho] + K(\rho) + \delta K(\rho)$$

Random initial condition

# Semiclassical version for approaches in Heavy-Ion collisions




*Adapted from J. Randrup et al, NPA538 (92).*

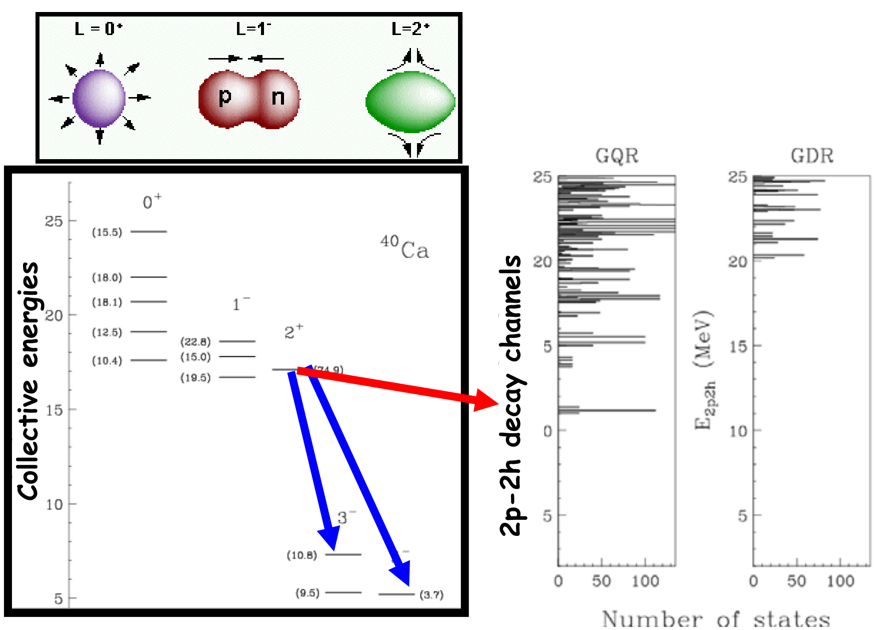
# Application in quantum systems

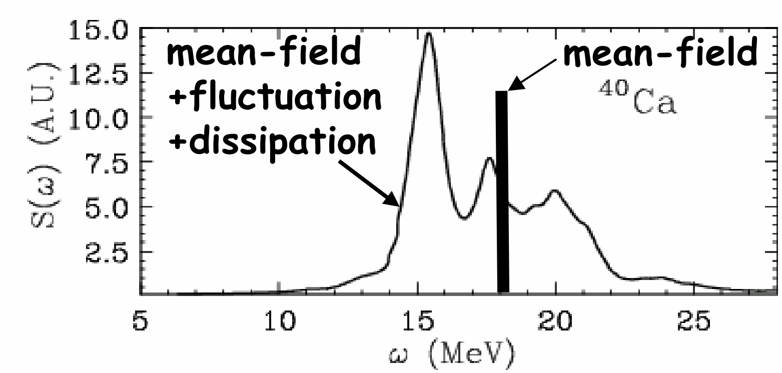
$$i\hbar\frac{\partial}{\partial t}\rho^{(n)} - [h(\rho^{(n)}), \rho^{(n)}] = K_I(\rho^{(n)}) + \delta K^{(n)}(t)$$

RPA

Coupling to 2p2h states

Coupling to ph-phonon




*D. Lacroix et al, Progress in Part. and Nucl. Phys. (2004)*

# Alternative formulation with Stochastic Schroedinger equations

GOAL: Restarting from an uncorrelated state $D = |\Phi_0\rangle\langle\Phi_0|$ we should:

1-have an estimate of $D = |\Psi(t)\rangle\langle\Psi(t)|$

2-interpret it as an average over jumps between "simple" states

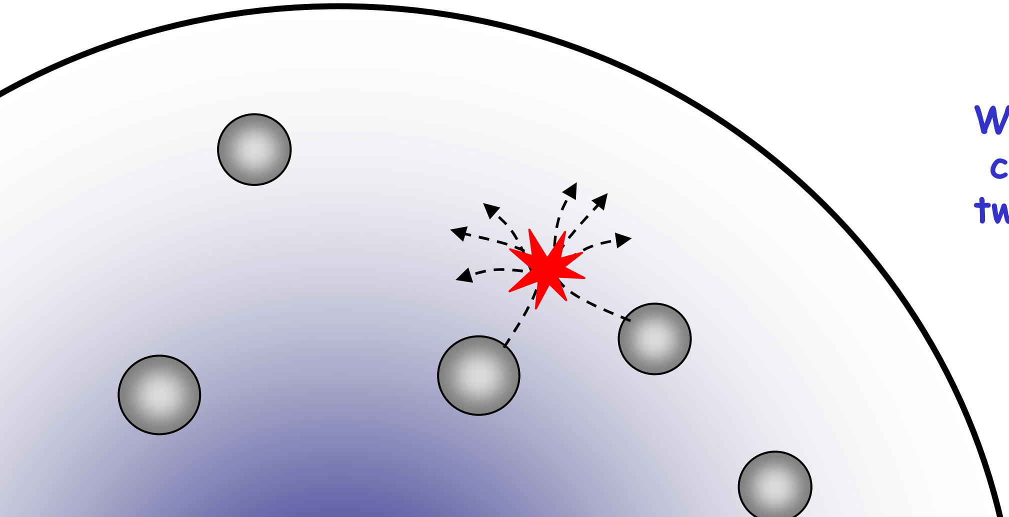
## Weak coupling approximation : perturbative treatment

*R.-G. Reinhard and E. Suraud, Ann. of Phys. 216, 98 (1992)*

$$|\Psi(t')\rangle = |\Phi(t')\rangle - \frac{i}{\hbar}\int \delta v_{12}(s)\,|\Phi(s)\rangle\, ds - \frac{1}{2\hbar^2} T\left(\int\int \delta v_{12}(s)\delta v_{12}(s')\, ds ds'\right)|\Phi(s)\rangle$$

Residual interaction in the mean-field interaction picture

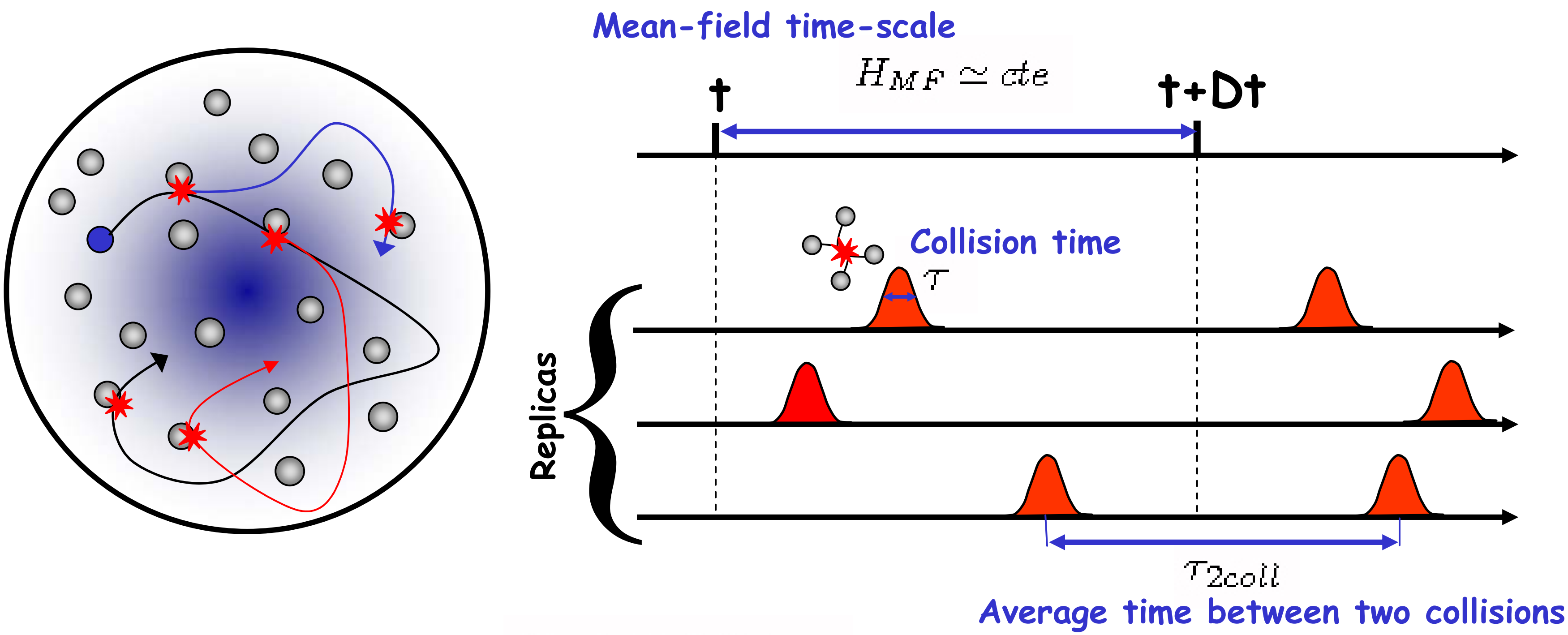
## Statistical assumption in the Markovian limit :



We assume that the residual interaction can be treated as an ensemble of two-body interaction:

$$\begin{cases} \overline{\delta v_{12}(s)} = 0 \\ \overline{\delta v_{12}(s)\delta v_{12}(s')} \propto \overline{\delta v_{12}^2(s)}\, e^{-(s-s')^2/2\tau^2} \end{cases}$$

# Time-scale and Markovian dynamics




Hypothesis : $\tau \ll \Delta t \ll \tau_{2coll}$

Two strategies can be considered:

- Considering waves directly
  (philosophy of exact treatment)

$$\overline{\Delta\left|\Psi\right\rangle} = \frac{\Delta t}{i\hbar} H_{MF}\left|\Phi(t)\right\rangle - \frac{\tau \Delta t}{2\hbar^2}\,\overline{\delta v_{12}^2}\left|\Phi(t)\right\rangle$$

- Considering densities directly
  (philosophy of dissipative treatment)

$$\overline{\Delta D} = \frac{\Delta t}{i\hbar}[H_{MF}, D] - \frac{\tau \Delta t}{2\hbar^2}\overline{[\delta v_{12}, [\delta v_{12}, D]]}$$

# Simplified scenario for introducing fluctuations beyond Mean-field

**Interpretation of the equation on waves as an average over jumps:**

$$\overline{\Delta\,|\Psi\rangle} = \frac{\Delta t}{i\hbar} H_{MF}\,|\Phi(t)\rangle - \frac{\tau \Delta t}{2\hbar^2}\,\overline{\delta v_{12}^2}\,|\Phi(t)\rangle \quad \Longleftrightarrow \quad \Delta\,|\Psi\rangle = \left\{ \frac{\Delta t}{i\hbar} H_{MF} + \Delta B \delta v_{12} + \frac{1}{2}\left(\Delta B \delta v_{12}\right)^2 \right\} |\Phi(t)\rangle$$

Let us simply assume that $\delta v_{12} \longrightarrow \sigma \delta v_{12}$ with $\Delta B = i\sigma \frac{\sqrt{\tau \Delta t}}{\hbar}$

## SSE in one-body space

**Assuming** $D_{ab} = |\Phi_a\rangle \langle \Phi_b|$ with $\langle \Phi_b \mid \Phi_a \rangle = 1$

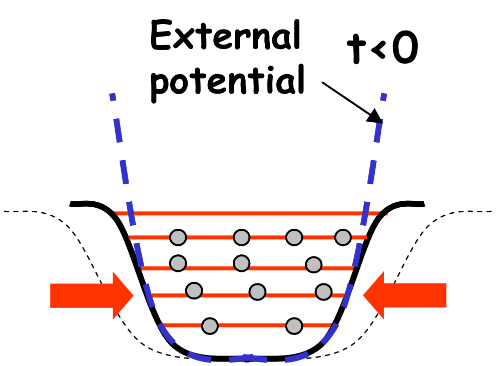
**and** $\langle a_i^+ a_j \delta v_{12}^2 \rangle \simeq \langle a_i^+ a_j \rangle \langle \delta v_{12}^2 \rangle + 2 \langle a_i^+ a_j \delta v_{12} \rangle \langle \delta v_{12} \rangle - 2 \langle a_i^+ a_j \rangle \langle \delta v_{12} \rangle^2$

$$d\rho = \frac{dt}{i\hbar}\left[h_{MF}, \rho\right] + dB_a (1-\rho) U_\delta(\rho) \rho + dB_b^* \rho U_\delta(\rho)(1-\rho)$$

*D. Lacroix, PRC73 (2006)*

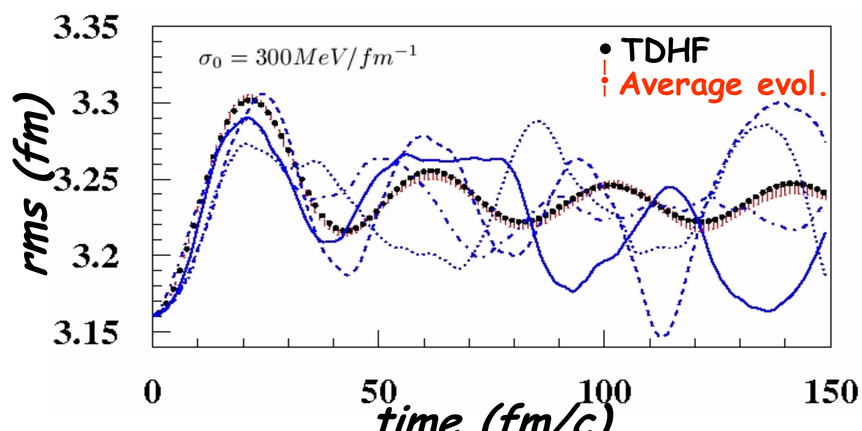
## Application

*Monopole vibration in $^{40}Ca$*



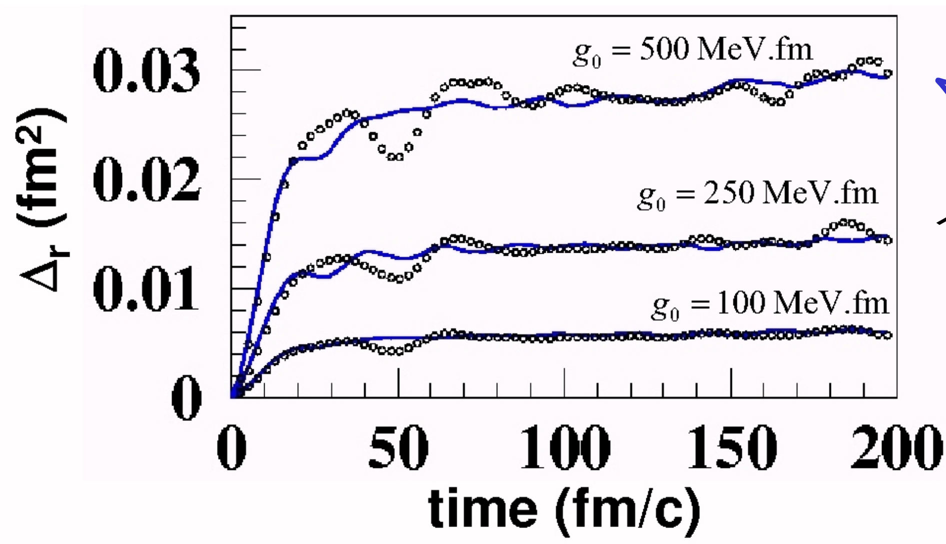

**Stochastic part:**

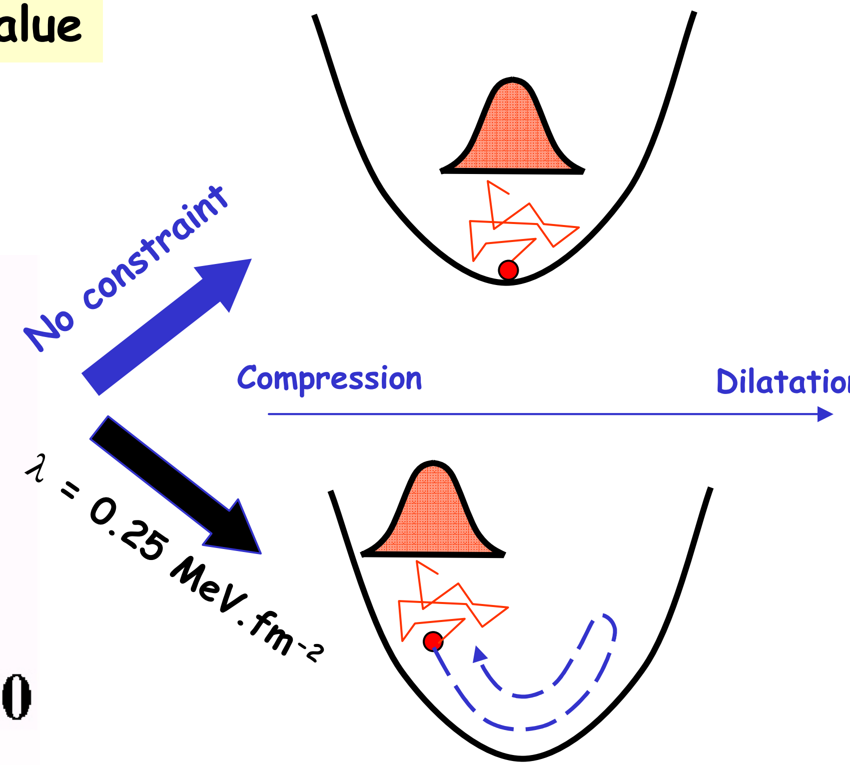
$$\delta v_{12} = \sigma_0 \delta(\mathbf{r}_1 - \mathbf{r}_2)$$



## Diffusion of the rms around the mean value

Standard deviation $\Delta_r = \sqrt{\overline{\langle r^2 \rangle^2} - \overline{\langle r^2 \rangle}^2}$




Similar to Nelson stochastic theory

*Nelson, Phys. Rev. 150, 1079 (1966).*

*Ruggiero and Zannetti, PRL 48, 963 (1982).*

## Summary and Critical discussion on the simplified scenario

- The stochastic method is directly applicable to nuclei
- It provide an easy way to introduce fluctuations beyond mean-field
- It does not account for dissipation.
- In nuclear physics the two particle-two-hole components dominates the residual interaction, but $U_{\delta_{2p2h}}(\rho) = 0$ !!!

# Quantum jump with dissipation: link between Extended TDHF and Lindblad eq.

**One-body density Master equation step by step**

$$\overline{\Delta D} = \frac{\Delta t}{i\hbar}[H_{MF}, D] - \frac{\tau \Delta t}{2\hbar^2}\overline{[\delta v_{12}, [\delta v_{12}, D]]}$$

**Initial simple state**

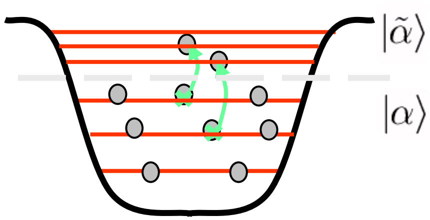
$$D = |\Phi\rangle\langle\Phi|$$

$$\rho = \sum_\alpha |\alpha\rangle\langle\alpha|$$

$$i\hbar\frac{d}{dt}\rho = [h_{MF}, \rho] - \frac{\tau}{2\hbar^2}\mathcal{D}(\rho)$$

with $\langle j|\mathcal{D}|i\rangle = \overline{\left\langle\left[\left[a_i^+ a_j, \delta v_{12}\right], \delta v_{12}\right]\right\rangle}$

**2p-2h nature of the interaction**



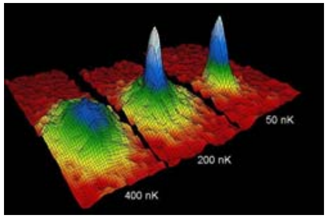

$$\mathcal{D}(\rho) = Tr_2\left[v_{12}, C_{12}\right]$$

with
$$C_{12} = (1-\rho_1)(1-\rho_2)v_{12}\rho_1\rho_2 - \rho_1\rho_2 v_{12}(1-\rho_1)(1-\rho_2)$$

**Separability of the interaction** $v_{12} = \sum_\lambda O_\lambda(1)O_\lambda(2)$

$$\mathcal{D}(\rho) = \sum_k \gamma_k \left(A_k A_k \rho + \rho A_k A_k - 2A_k\rho A_k\right)$$

- Dissipation contained in Extended TDHF is included
- The master equation is a Lindblad equation
- Associated SSE *D. Lacroix, PRC73 (2006)*




# Application to Bose condensate

1D bose condensate with gaussian two-body interaction

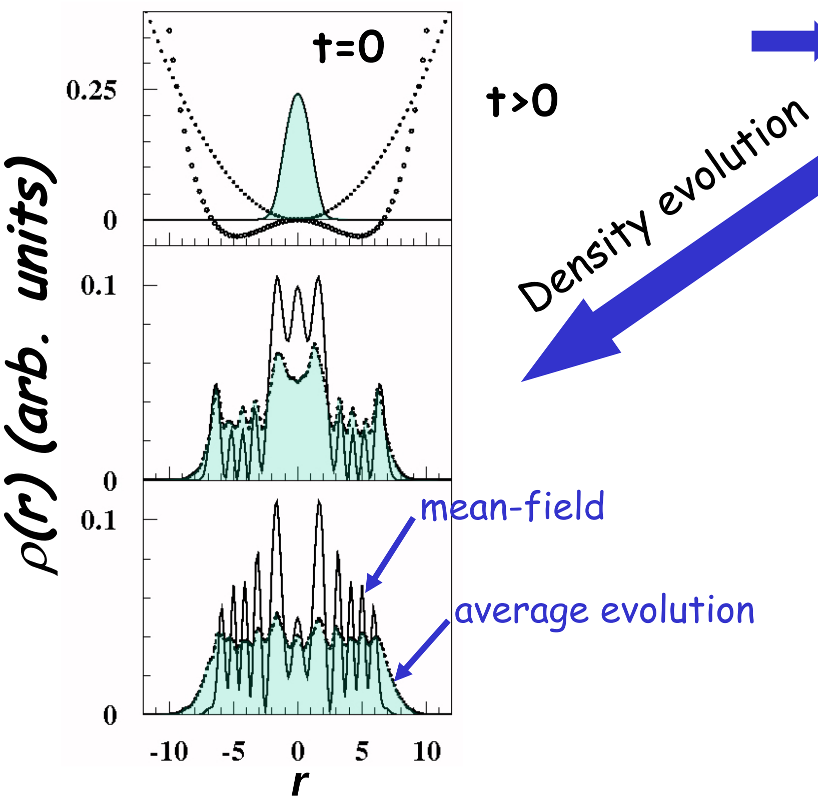
N-body density: $D = |N:\alpha\rangle\langle N:\alpha|$

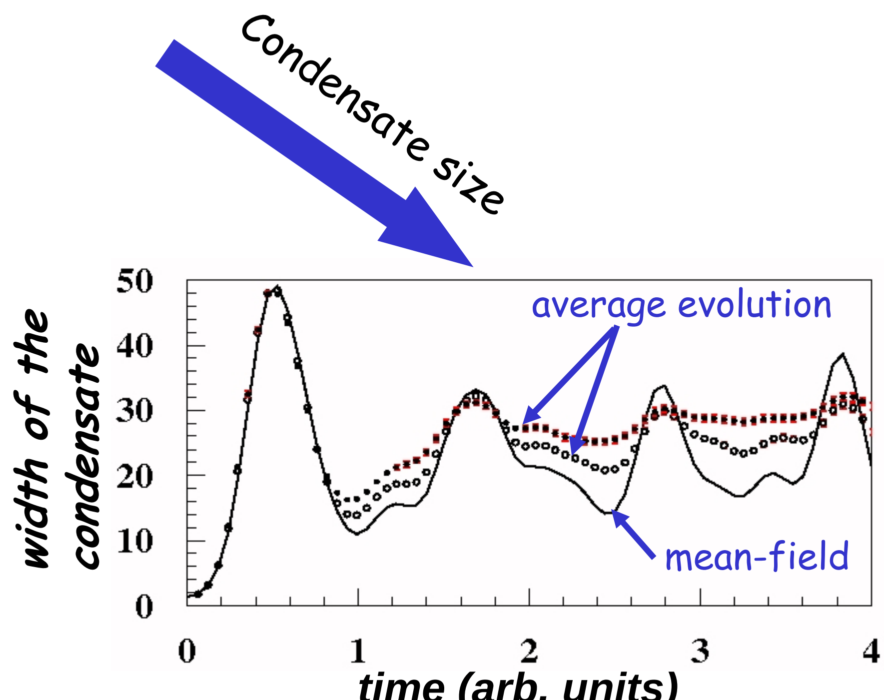
SSE on single-particle state :

$$d|\alpha\rangle = \left\{ \frac{dt}{i\hbar} h_{MF}(\rho) + \sum_k dW_k (1-\rho) A_k - \frac{dt\tau}{2\hbar^2} \sum_k \gamma_k \left[ A_k^2 \rho + \rho A_k \rho A_k - 2 A_k \rho A_k \right] \right\} |\alpha\rangle$$

with $dW_k dW_{k'} = -\frac{dt\tau}{\hbar^2}\gamma_k \delta_{kk'}$

→ The numerical effort is fixed by the number of $A_k$



# Summary

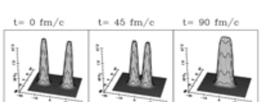
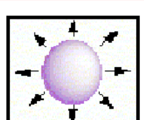
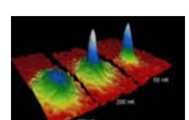
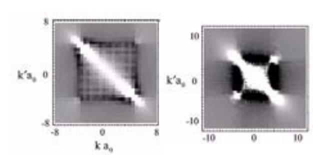
## Quantum Jump (QJ) methods (or SSE) to extend mean-field

**Approximate evolution**

| Mean-field | Simplified QJ | Generalized QJ | Exact QJ | variational QJ |
|---|---|---|---|---|
| $D = \|\Phi\rangle\langle\Phi\|$ | $D = \overline{\|\Phi_1\rangle\langle\Phi_2\|}$ | $D = \overline{\|\Phi\rangle\langle\Phi\|}$ | $D = \overline{\|\Phi_1\rangle\langle\Phi_2\|}$ | $D = \overline{\|Q_1\rangle\langle Q_2\|}$ <br> $\|Q_1\rangle = \|q_1, \cdots, q_N\rangle$ |
| ~~Fluctuation~~ <br> ~~Dissipation~~ | Fluctuation ✓ <br> ~~Dissipation~~ | Fluctuation ✓ <br> Dissipation ✓ | Everything ✓ | Partially everything ✓ |

## Numerical issues

| | Flexible | Fixed | Fixed | Flexible |
|---|---|---|---|---|

# Some issues

- Stochastic mean-field in the Density Functional context?

- Link with other stochastic theories?
  - -path integrals
  - -Nelson theory
  - -measurement theory (non-demolition, conditional)…
  - -Quantum Monte-Carlo methods

- Stochastic mean-field as an alternative to Generator Coordinate Method?

- Solution to numerical instabilities

- Dissipative system: role of non-Markovian effects?

…