Méthodes Spectrales pour la Propagation et la Quantification d'Incertitudes Applications aux Écoulements

Olivier Le Maître[1]

[1]olm@limsi.fr
Laboratoire d'Informatique pour la Mécanique et les Sciences de l'Ingénieur
UPR-3251
Orsay.

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Outline

Simulation framework.

Basic ingredients

- Understanding of the physics involved (optional ?) :
  selection of the **mathematical model**.
- **Numerical method**(s) to solve the model.
- Specify a set of **data** :
  select a system among the class spanned by the model.

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Simulation errors

- **Model errors** : physical approximations and simplifications.
- **Numerical errors** : discretization, approximate solvers, finite arithmetics.
- **Data error** : boundary/initial conditions, model constants and parameters, external forcings, ...

Sources of data uncertainty

- Inherent **variability** (*e.g.* industrial processes).
- **Epistemologic** uncertainty (*e.g.* model constants).
- **May not be fully reductible, even theoretically**.

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Probabilistic framework

- Define an abstract probability space $(\Omega, \mathcal{A}, d\mu)$.
- Consider **data** D as **random** quantity : $D(\omega), \ \omega \in \Omega$.
- **Simulation output** S is **random** and on $(\Omega, \mathcal{A}, d\mu)$.

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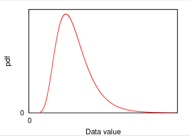
- **Simulation output** S is **random** and on $(\Omega, \mathcal{A}, d\mu)$.
- Data D and simulation output S are dependent random quantities (through the mathematical model $\mathcal{M}$) :

$$\mathcal{M}(S(\omega), D(\omega)) = 0, \quad \forall \omega \in \Omega.$$

Propagation and Quantification of data uncertainty

Data density



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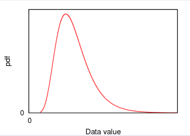

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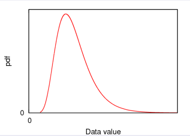

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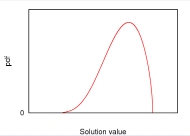

Propagation and Quantification of data uncertainty

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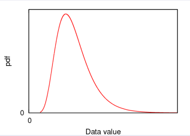

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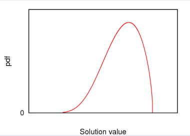

- **Variability** in model output : numerical error bars.
- Assessment of **predictability**.
- Support **decision making process**.
- What type of information (abstract quantities, confidence intervals, density estimations, structure of dependencies, ...) one needs ?

Deterministic methods

- Sensitivity analysis (adjoint based, AD, . . .) : local.
- Perturbation techniques : limited to low order and simple data uncertainty.
- Neuman expansions : limited to low expansion order.
- Moments method : closure problem (non-Gaussian / non-linear problems).

Simulation techniques Monte-Carlo

Spectral Methods

Deterministic methods

Simulation techniques — Monte-Carlo

- Generate a sample set of data realizations and compute the corresponding sample set of model ouput.
- Use sample set based random estimates of abstract characterizations (moments, correlations, . . .).
- Plus : **Very robust** and re-use **deterministic codes** : (parallelization, complex data uncertainty).
- Minus : **slow convergence of the random estimates** with the sample set dimension.

Spectral Methods

Deterministic methods

Simulation techniques — Monte-Carlo

Spectral Methods

- Parametrization of the data with random variables (RVs).
- ⊥ projection of solution on the space spanned by the RVs.
- Plus : **arbitrary level of uncertainty, deterministic approach, convergence rate, information contained**.
- Minus : **parametrizations** (limited # of RVs), **adaptation of simulation tools** (legacy codes), **robustness** (non-linear problems, non-smooth output, …).
- **Constant developments and improvements** (be faithfull !).

Outline

Polynomial Chaos expansion Wiener-1938

Any well behaved RV $\theta(\omega)$ (*e.g.* 2nd order one) defined on $(\Omega, \mathcal{A}, d\mu)$ has a **convergent expansion** of the form :

$$\begin{aligned}\theta(\omega) &= u_0\Gamma_0 + \sum_{i_1=1}^{\infty} \theta_{i_1}\Gamma_1(\xi_{i_1}(\omega)) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} \theta_{i_1,i_2}\Gamma_2(\xi_{i_1}(\omega), \xi_{i_2}(\omega)) \\ &+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} \theta_{i_1,i_2,i_3}\Gamma_3(\xi_{i_1}(\omega), \xi_{i_2}(\omega), \xi_{i_3}(\omega)) + \dots\end{aligned}$$

- $\{\xi_1, \xi_2, \dots\}$: **independent normalized Gaussian RVs.**
- Γ_p **polynomials with degree** p, orthogonal to $\Gamma_q, \forall q < p$.
- Convergence in the **mean square sense** (Cameron and Martin, 1947).

Polynomial Chaos expansion — Wiener-1938

Truncated PC expansion at order p and n RVs :

$$\theta(\omega) \approx \sum_{k=0}^{\mathrm{P}} \theta_k \Psi_k(\boldsymbol{\xi}(\omega)), \quad \boldsymbol{\xi} = \{\xi_1, \dots, \xi_n\}, \quad \mathrm{P} = \frac{(n+p)!}{n!p!}.$$

- $\{\theta_k\}_{k=0,\dots,\mathrm{P}}$: **deterministic** expansion coefficients,
- $\{\Psi_k\}_{k=0,\dots,\mathrm{P}}$: $\perp$ **random polynomials** for the **inner product** defined with the density of ξ as weight :

$$\begin{aligned} \langle \Psi_k \Psi_l \rangle &\equiv \int_\Omega \Psi_k(\boldsymbol{\xi}(\omega)) \Psi_l(\boldsymbol{\xi}(\omega)) d\mu(\omega) \\ &= \int \Psi_k(\xi) \Psi_l(\xi) p(\xi) d\xi = \delta_{kl} \left\langle \Psi_k^2 \right\rangle . \end{aligned}$$

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- $p(\xi) = \prod_{i=1}^{n} \frac{\exp\left(-\xi_i^2/2\right)}{\sqrt{2\pi}} \Longrightarrow \Psi_k(\xi)$: **Hermite polynomials**

Polynomial Chaos expansion — Wiener-1938

Truncated PC expansion : $\theta(\omega) \approx \sum_{k=0}^{P} \theta_k \Psi_k(\xi(\omega))$.

- Convention $\Psi_0 \equiv 1$: mean mode.
- **Expectation** of θ :

$$E[\theta] \equiv \int_{\Omega} \theta(\omega) d\mu(\omega) \approx \sum_{k=0}^{P} \theta_k \int_{\Omega_{\boldsymbol{\xi}}} \Psi_k(\xi) p(\xi) d\xi = \theta_0.$$

- **Variance** of θ :

$$V[\theta] = E[\theta^2] - E[\theta]^2 \approx \sum_{k=1}^{P} \theta_k^2 \left\langle \Psi_k \Psi_k \right\rangle .$$

- Extension to **random vectors & stochastic processes** :

$$\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_m \end{pmatrix} (\omega, \boldsymbol{x}, t) \approx \sum_{k=0}^{P} \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_m \end{pmatrix}_k (\boldsymbol{x}, t)\, \Psi_k(\xi(\omega)).$$

Generalized PC expansion — Xiu and Karniadakis-2002

Askey scheme

Distribution of ξ_i	Polynomial familly
Gaussian	Hermite
Uniform	Legendre
Exponential	Laguerre
β-distribution	Jacobi

Also : discrete RVs (Poisson process).

$$\theta(\omega) \approx \sum_{k=0}^{P} \theta_k \Psi_k(\xi(\omega))$$

where Ψ_k : classical (or mixture of) polynomials.

Data parametrization

Parametrization of D using $\mathrm{N} < \infty$ independent RVs with prescribed distribution $p(\xi)$:

$$D(\omega) = D(\boldsymbol{\xi}(\omega)), \quad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_{\mathrm{N}}) \in \Omega_\xi.$$

- Transformation of random variables : $D(\omega)$ RV.
- Karhunen-Loève expansion : $D(\boldsymbol{x}, \omega)$ stochastic process.
- Independent components analysis.

Model

Solution expansion

Data parametrization

Model

We assume that $\forall \xi(\omega) \in \Omega_\xi$, the problem $\mathcal{M}(S, D(\xi(\omega)) = 0$

1. is **well-posed**,
2. has a **unique solution**, denoted $S(\xi(\omega))$,

and that the random solution $\boxed{S \in L_2(\Omega_\xi, p_\xi)}$:

$$\left\langle S^2 \right\rangle = \int_\Omega S^2(\xi(\omega)) d\mu(\omega) = \int_{\Omega_\xi} S^2(\xi) p(\xi) d\xi < +\infty.$$

Solution expansion

Data parametrization

Model

Solution expansion

Let $\{\Psi_0, \Psi_1, \ldots\}$ be a Hilbert basis of $L_2(\Omega_\xi, p_\xi)$ then

$$S(\xi(\omega)) = \sum_k S_k \Psi_k(\xi(\omega)).$$

- Knowledge of the **spectral coefficients** S_k fully **determine the random solution**.
- Makes explicit the dependence between $D(\xi)$ and $S(\xi)$.

Data parametrization

Model

Solution expansion

Let $\{\Psi_0, \Psi_1, \dots\}$ be a Hilbert basis of $L_2(\Omega_\xi, p_\xi)$ then

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- Knowledge of the **spectral coefficients S_k** fully **determine the random solution**.
- Makes explicit the dependence between $D(\xi)$ and $S(\xi)$.
- **Need efficient procedure to compute the S_k.**

Non intrusive techniques

- Compute/estimate spectral coefficients *via* a set of deterministic model solutions.
- Requires a deterministic solver only.
- Overcome issues related to non-linearities.
- Suffers from the curse of dimensionnality.

NI-techniques

Galerkin projection

- Weak solution of the stochastic problem $\mathcal{M}(S, D) = 0$.
- Needs adaptation of deterministic codes.
- Usually more efficient than NI techniques.
- Better suited to improvement (error estimate, optimal and basis reduction, ...), thanks to spectral theory and functional analysis.

Galerkin projection **Method of weighted residual**

① Introduce truncated expansions in model equations.

② Require residual to be $\perp$ to the subspace.

$$\left\langle \mathcal{M}\left(\sum_{k=0}^{\mathrm{P}} S_k \Psi_k(\xi), D(\xi)\right) \Psi_m(\xi)\right\rangle = 0 \quad \text{for } m = 0, \ldots, \mathrm{P}.$$

Set of $P+1$ **coupled** problems.

Plus

- Implicitly account for modes' coupling.
- Often inherit properties of the deterministic model.

Minus

- Requires adaptation of deterministic solvers.
- Treatment of non-linearities.

Outline

Examples of Application to Fluid Flows

- Natural convection : Boussinesq approximation Goto example
- Natural convection : Low-Mach approximation Goto example
- Electrophoresis : coupled physical problems Goto example
- Lagrangian formulation : particle method Goto example

Outline

Motivations

GPC expansions fail for some problems because of : ▸ Example

① **Non-linearities** requiring large polynomial orders for global approximation over uncertainty range.

② **Non-smooth or steep dependences** of the solution w.r.t. the uncertain data (*e.g.* **parametric bifurcations**, absolute value, threshold effect, . . .).

③ Oscillating character of the polynomials.

Response : Le Maître et al, JCPs (2004).

Wiener-type orthogonal expansion (multiwavelets) using **Multi-Resolution-Analysis** .

- ✓ Piecewise polynomial.
- ✓ Convergence in polynomial order **and** resolution level.
- ✓ Discontinuous dependences.
- ✓ Local control of the resolution.
- ✓ Adaptive strategy.

Multi-resolution space

For $\mathrm{No} = 0, 1, \ldots$ and $k = 0, 1, \ldots$, $\mathbf{V}_k^{\mathrm{No}}$ is the space of **piecewise polynomial functions** $f : x \in [-1, 1] \mapsto \mathbb{R}$:

$$\mathbf{V}_k^{\mathrm{No}} \equiv \Big\{ f : \text{ the restriction of } f \text{ on } (2^{-k} l, 2^{-k}(l+1)) \in \mathbb{P}_{\mathrm{No}} \quad \text{for } l = 0, \ldots, 2^k - 1 \Big\},$$

where $\mathbb{P}_{\mathrm{No}}$ is the space of polynomials with degree $\leq \mathrm{No}$.
We have :

- $Dim(\mathbf{V}_k^{\mathrm{No}}) = (\mathrm{No} + 1)(2^k)$,
- $\mathbf{V}_0^{\mathrm{No}} \subset \mathbf{V}_1^{\mathrm{No}} \subset \cdots \subset \mathbf{V}_k^{\mathrm{No}} \subset \ldots$
- $\mathbf{V}^{\mathrm{No}} \equiv \overline{\bigcup}_{k \geq 0} \mathbf{V}_k^{\mathrm{No}}$ is dense in $L_2([0, 1])$ with the scalar product

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx.$$

Multi-wavelet space

Let us denote $\mathbf{W}_k^{\mathrm{No}}$, $k = 0, 1, 2, \ldots,$ the orthogonal complement of $\mathbf{V}_k^{\mathrm{No}}$ in $\mathbf{V}_{k+1}^{\mathrm{No}}$:

$$\mathbf{V}_k^{\mathrm{No}} \oplus \mathbf{W}_k^{\mathrm{No}} = \mathbf{V}_{k+1}^{\mathrm{No}}, \quad \mathbf{W}_k^{\mathrm{No}} \perp \mathbf{V}_k^{\mathrm{No}},$$

so

$$\mathbf{V}_0^{\mathrm{No}} \bigoplus_{k \geq 0} \mathbf{W}_k^{\mathrm{No}} = L^2([0, 1]).$$

Let $\{\psi_0, \psi_1, \ldots, \psi_{\mathrm{No}}\}$ be an orthonormal basis of $\mathbf{W}_0^{\mathrm{No}}$:

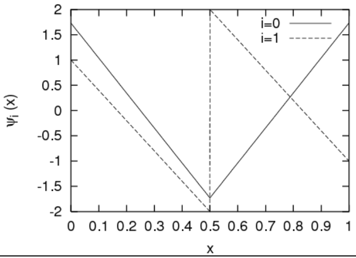
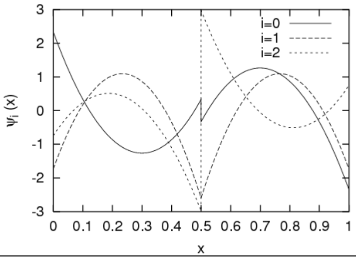
$$\langle \psi_i(x), \psi_j(x) \rangle = \delta_{ij},$$

and since $\mathbf{W}_0^{\mathrm{No}} \perp \mathbf{V}_0^{\mathrm{No}}$ we have

$$\left\langle \psi_j, x^i \right\rangle = 0, \quad 0 \leq i, j \leq \mathrm{No}.$$

Multi-wavelet space

The ψ_j are the **generating functions** of the MRA system.

$\mathrm{No} = 1$	$\mathrm{No} = 2$
	

Multi-wavelet space

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Multi-wavelets ψ_{jl}^k

$$\psi_{jl}^k(x) = 2^{k/2}\psi_j(2^k x - l), \quad j = 0, \dots, \mathrm{No}, \text{ and } l = 0, \dots, 2^k - 1.$$

- $\mathrm{Supp}(\psi_{jl}^k) = [2^{-k} l, 2^{-k}(l+1)]$.
- $\left\langle \psi_{il}^k, \psi_{jm}^{k'} \right\rangle = \delta_{ij}\delta_{lm}\delta_{kk'}$.

Basis of V_0^{No} — Legendre polynomials

$$\phi_i(x) = \frac{\mathcal{L}e_i(2x-1)}{L_i}, \quad i = 0, 1, \dots, \mathrm{No},$$

$$\left\langle \phi_i(x), \phi_j(x) \right\rangle = \delta_{ij} \text{ for } i, j = 0, \dots, \mathrm{No}.$$

Projection on $\mathbf{V}_{\mathrm{Nr}}^{\mathrm{No}}$

Let us denote $f^{\mathrm{No},\mathrm{Nr}}$ the projection of f on $\mathbf{V}_{\mathrm{Nr}}^{\mathrm{No}}$:

$$f^{\mathrm{No},\mathrm{Nr}}(x) \equiv \mathcal{P}_{\mathrm{Nr}}^{\mathrm{No}}[f] = \sum_{i=0}^{\mathrm{No}} f_i \psi_i(x) + \sum_{k=0}^{\mathrm{Nr}-1} \sum_{l=0}^{2^k-1} \left(\sum_{i=0}^{\mathrm{No}} \delta f_{il}^k \psi_{il}^k(x) \right),$$

where

$$f_i = \langle f, \phi_i \rangle, \text{and } \delta f_{il}^k = \left\langle \left\{ \mathcal{P}_{k+1}^{\mathrm{No}}[f] - \mathcal{P}_k^{\mathrm{No}}[f] \right\}, \psi_{il}^k \right\rangle.$$

For $f \in L_2([0,1])$, the projection error can be made arbitrarily small by increasing the expansion order No and/or resolution level Nr.

Application of MRA to UQ

One-dimensional case

- $\xi(\omega)$: RV with density $\mathrm{pdf}(\xi)$, CDF $q(\xi) = \int_{-\infty}^{\xi} \mathrm{pdf}(\xi') d\xi'$.
- $\theta(\xi) \in L_2(\Omega_\xi)$.
- $\theta(\xi) = \theta(q^{-1}(x)) = \widetilde{\theta}(x)$ for $x \sim U(0,1)$.
- $\widetilde{\theta}(x) \in L_2([0,1])$.

$$\widetilde{\theta}(x(\omega)) \approx \sum_k \widetilde{\theta}_k W_k(x(\omega)), \quad x \sim U(0,1),$$

W_k elements of the MRA system.

Application of MRA to UQ

N-dimensionnal case

- Proceed by **tensorization** of 1-D MRA system.
- $\widetilde{\theta}(\boldsymbol{x}) \equiv \widetilde{\theta}(x_1, \ldots, x_{\mathrm{N}}) \approx \sum_{\boldsymbol{k}} \widetilde{\theta}_{\boldsymbol{k}} \mathcal{M}^w{}_{\boldsymbol{k}}(x_1, \ldots, x_{\mathrm{N}})$.
- $\mathcal{M}^w{}_{\boldsymbol{k}}(\boldsymbol{x}) = W_{k_1}(x_1) \times \cdots \times W_{k_{\mathrm{N}}}(x_{\mathrm{N}})$.

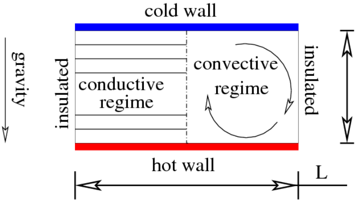
Summary

- Expansion in terms of **CDF** of random parameters.
- **Piecewise polynomial approximation**.
- Error reduction through p (No) or h (Nr) **refinement**.
- **Fast increase with No, Nr and N of approximation space's dimension** (calls for adaptive techniques).

Example

Rayleigh-Bénard Instability

- Aspect ratio : $A = L/H = 2$;
- Prandtl number : $\mathrm{Pr} = \frac{\mu C_p}{\kappa} = 0.7$;
- Rayleigh number : $\mathrm{Ra} = \frac{\rho g \beta \Delta T H^3}{\mu \kappa}$.



Model : **Boussinesq equations**.

Parameter and uncertainty :

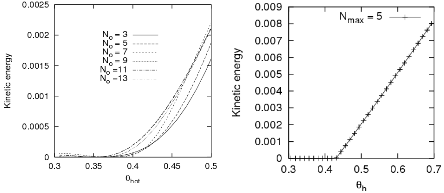
- $\mathrm{Ra} = 2150$ (slightly above critical)
- $\Theta_{\mathrm{hot}}(\xi) = \frac{1}{2} + 0.2\xi, \quad \xi$ U.D. in $[-1, 1]$

Both conductive **and** convective regimes are explored.

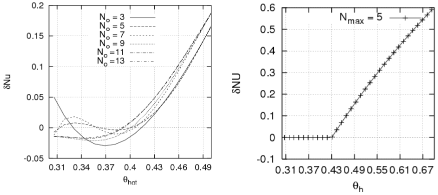
The process has a **discontinuity** in the uncertainty range.

Comparison Legendre / Wiener-Haar ($\mathrm{Nr} = 5$) solutions.

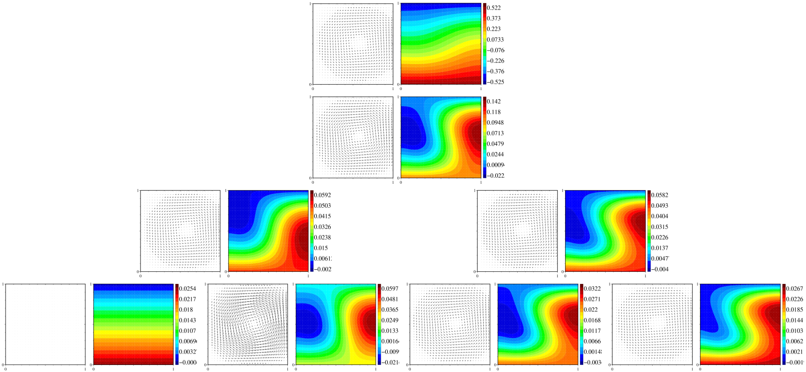
Kinetic energy as function of $\theta_{hot}(\xi)$.



Heat transfer enhancement as a function of $\theta_{hot}(\xi)$.



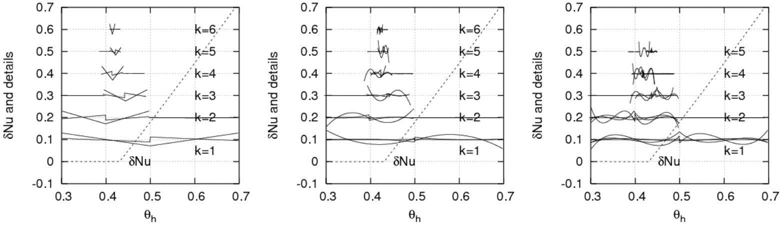
Hierarchy of velocity and temperature modes (Wiener-Haar)



Cold Hot

Heat-transfer enhancement (from conduct. solution).

Adaptive MRA scheme for $\mathrm{No} = 1, 2$ and 3



Only details around critical points are computed.

Limitations of MRA

- ✔ Fast increase of the **basis dimension** with No and resolution level Nr.
- ✔ **Adaptivity** possible but quickly **cumbersome** with increasing N(number of stochastic dimensions).

A More efficient approach

Remark : Spectral problems present **no differential operator along stochastic dimensions**. (Model solutions for different data are independent)

- ⇨ Strongly suggests a **domain decomposition** technique in the parameter space $\Omega_\xi = [0, 1]^{\mathrm{N}}$.

Partition of the random parameter space

Domain decomposition.

Basic principle : **zooming.**

- ✔ Define a generic expansion basis for $[0,1]^{\mathrm{N}}$:

  N-Dimensional Legendre basis

  +

  1-D first resolution level Multi-Wavelets.

- ✔ Rescale and translate this basis to **expand locally** the solution on non-overlapping sub-domains $\Omega_i \subset [0,1]^{\mathrm{N}}$.
- ✔ Decide if the expansion is sufficient over Ω_i ; If not :

  break it into smaller sub-domains

  along under-resolved dimensions only

- ✔ **Refinement strategy based on 1-D details.**

Reaction surface problem $\text{No} = 3$

Convergence with ϵ_r :

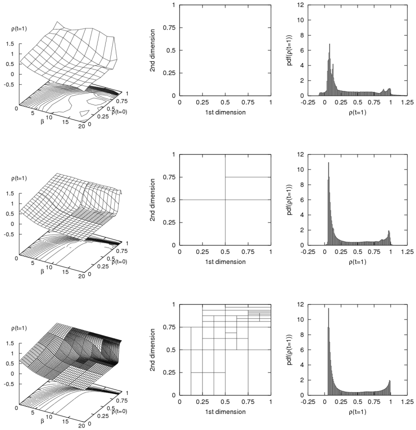
Governing equations :

$$\begin{cases} \dfrac{d\rho}{dt} = \alpha(1-\rho) - \gamma\rho \\ \qquad -\beta(\rho-1)\rho^2 \\ \rho(t=0) = \rho_0 \end{cases}$$

Uncertainty

- ✓ $\rho_0 \sim U(0,1)$.
- ✓ $\beta \sim U(0,20)$.
- ✓ ρ_0, β ind. RVs.

$\Rightarrow$ 2 Stochastic dim.



Applied for up to 8 stochastic dimensions and a complex chemical mechanism (Le Maître *et al*, J. Sci. Comp. 2007.)

A posterior error estimation (Lionel Mathelin, LIMSI-CNRS)

Objective : design less heuristic criteria / error indicator.

Variational framework

Solve for $U(\boldsymbol{x} \in \Omega_x, \xi \in \Omega_\xi) \in \mathcal{V}_x \otimes \mathcal{V}_\xi$

$$A(U; \Phi | D) = B(\Phi | D) \quad \forall \Phi \in \mathcal{V}_x \otimes \mathcal{V}_\xi,$$

where :

- $\mathcal{V}_x$ suitable deterministic Hilbert space,
- $\mathcal{V}_\xi \equiv L_2(\Omega_\xi, p_\xi)$ space of 2nd order RV,
- $A(U; \Phi | D) = \int_{\Omega_\xi} a(U(\xi); \Phi(\xi) | D(\xi)) p_\xi(\xi) d\xi$,
- $B(\Phi | D) = \int_{\Omega_\xi} b(\Phi(\xi) | D(\xi)) p_\xi(\xi) d\xi$,,
- $a(.;.|.)$ a deterministic semi-linear form,
- $b(.|.)$ a linear form.

Deterministic finite element space

- $\Omega_x = \bigcup_{l=1}^{N_x} \Omega_x^{(l)}$.
- $U^h(x \in \Omega_x^{(l)}) = \sum_{i=1}^{\mathrm{Nd}(l)} U_i^{(l)} \mathcal{N}_i^{(l)}(x)$.

$$\mathcal{V}_x^h = \mathrm{span}\left(\{\mathcal{N}_i^{(l)}\},\ 1 \leq l \leq N_x,\ 1 \leq i \leq \mathrm{Nd}(l)\right).$$

Stochastic space

- $\Omega_\xi = \bigcup_{m=1}^{Nb} \Omega_\xi^{(m)}$,
- $\Omega_\xi^{(m)} = [\xi_1^{(m),-}, \xi_1^{(m),+}] \times \cdots \times [\xi_{\mathrm{N}}^{(m),-}, \xi_{\mathrm{N}}^{(m),+}]$,
- $U^h(\xi \in \Omega_\xi^{(m)}) = \sum_{k=0}^{\mathrm{P}(m)} u_k^{(m)} \Psi_k^{(m)}(\xi)$,

$$\mathcal{V}_\xi^h = \mathrm{span}\left(\left\{\Psi_k^{(m)}\right\},\ 1 \leq m \leq Nb,\ 0 \leq k \leq \mathrm{P}(m)\right)$$

Approximation space

$$\mathcal{V}^h = \mathcal{V}_x^h \otimes \mathcal{V}_\xi^h.$$

The approximate solution at point (x, ξ) of $\Omega \equiv \Omega_x \times \Omega_\xi$, is $U^h\left(x \in \Omega_x^{(l)}, \xi \in \Omega_\xi^{(m)}\right) = \sum_{i=1}^{\mathrm{Nd}(l)} \sum_{k=0}^{\mathrm{P}(m)} u_{i,k}^{(l,m)} \mathcal{N}_i^{(l)}(x)\, \Psi_k^{(m)}(\xi)$ and solves

$$A(U^h; \Phi^h | D^h) = B(\Phi^h | D^h) \quad \forall \Phi^h \in \mathcal{V}^h.$$

Error estimation

For $\mathcal{J} : \Omega_x \times \Omega_\xi \mapsto \mathbb{R}$, the approximation error is measured as

$$\eta = \left| \mathcal{J}(U) - \mathcal{J}\left(U^h\right) \right|.$$

The exact solution being unknown η has to be estimated.

Dual-based error estimate

$$\mathcal{J}(U) - \mathcal{J}(U^h) \approx B(\widetilde{Z} - Z^h | D^h) - A(U^h; \widetilde{Z} - Z^h | D^h),$$

where

- Z^h is the approximate dual solution satisfying
$$\mathcal{J}'(U^h; \Phi) - A'(U^h; \Phi, Z^h | D^h) = 0 \quad \forall \Phi \in \mathcal{V}^h,$$
- $\widetilde{Z} \in \mathcal{V}^{\widetilde{h}} \supset \mathcal{V}^h$ an estimate of the exact dual solution :
$$\mathcal{J}'(U^h; \widetilde{\Phi}) - A'(U^h; \widetilde{\Phi}, \widetilde{Z} | D^{\widetilde{h}}) = 0 \quad \forall \widetilde{\Phi} \in \mathcal{V}^{\widetilde{h}}.$$

In practice : $\mathcal{V}^{\widetilde{h}}$ is constructed by increasing the stochastic and finite element orders of $\mathcal{V}^h$.

Remark : Dual problems are linear, primes denote Gateau derivatives :

$$\mathcal{J}'(U, \Phi) = \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{J}(U) - J(U + \epsilon\Phi)}{\epsilon}.$$

Local error estimate

$$\eta = \left| \mathcal{J}(U) - \mathcal{J}(U^h) \right| \leq \sum_{l=1}^{N_x} \sum_{m=1}^{Nb} \eta_{l,m},$$

where $\eta_{l,m}$ is the local contribution of $\left(\Omega_x^{(l)} \times \Omega_\xi^{(m)} \right)$ to the aposteriori error estimation.

To ensure $\eta < \epsilon$, the approximation space $\mathcal{V}^h$ is refined such that

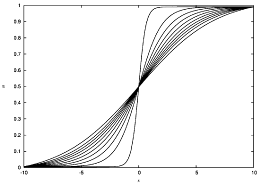
$$\eta_{l,m} < \frac{\epsilon_\eta}{N_x Nb} = \epsilon, \quad \forall l, m \in [1, N_x] \times [1, Nb].$$

Refinement scheme

- **Refine $\mathcal{V}_x$ or $\mathcal{V}_\xi$** ?
- **What type of refinement : h or p** ?
- If h_ξ, then **along which stochastic dimension(s)** ?

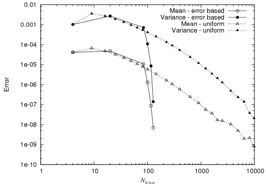
Example : Burger's equation with uncertain viscosity ($\mathrm{N} = 2$).

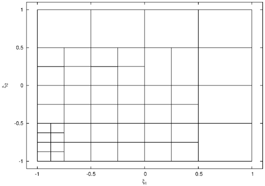
- $\mathcal{V}_x^h$: 10 spectral finite elements (order 15).
- Stochastic order $\mathrm{No} = 2$ with **isotropic h_ξ-refinement**.



$$u(\theta)\frac{\partial u(\theta)}{\partial x} = \nu(\theta)\frac{\partial^2 x(\theta)}{\partial x^2}.$$

Errors on computed mean and variance as a function of the number of primal and dual problems solved. Comparison of adaptive and uniform refinements.





(Mathelin and Le Maître, Com. Appl. Math and Comp., 2007)

Outline

1. **Introduction**
   - Simulation and errors
   - Data uncertainty
   - Alternative UQ methods
2. **Spectral UQ**
   - Generalized PC expansion
   - Application to spectral UQ
   - Solution Techniques
3. **Examples : Fluid flows**
4. **Advanced Topics**
   - Multi-resolution-analysis
   - Adaptive Techniques
   - A posterior error estimation
5. **Conclusive remarks**

Improvement of Spectral UQ

- Computational efficiency (steady-solvers, pre-conditioning, multigrid techniques,. . .).
- Development of directional error estimates to improve adaptive techniques.
- Construction of reduced basis.
- Adaptive non-intrusive technique.

Open problems

- Existence/treatment of multiple solutions !
- Stochastic eigen-value problems (many issues remaining to be addressed).
- . . .

Acknowlegments

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- Jean-Marc Martinez, CEA (saclay).

Natural convection — Boussinesq approximation

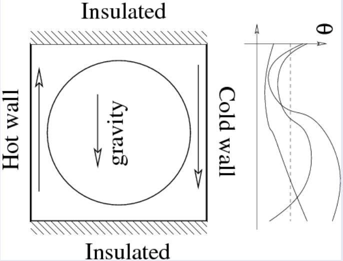
Governing equations

- **Momentum :** $$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{\mathrm{Pr}}{\sqrt{\mathrm{Ra}}} \nabla^2 \boldsymbol{u} + \mathrm{Pr}\theta \boldsymbol{y}$$
- **Mass :** $$\nabla \cdot \boldsymbol{u} = 0$$
- **Energy :** $$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta = \frac{1}{\sqrt{\mathrm{Ra}}} \nabla^2 \theta$$

Uncertain boundary conditions

Governing equations

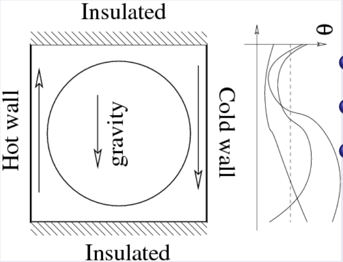
Uncertain boundary conditions



Natural convection — Boussinesq approximation

Governing equations

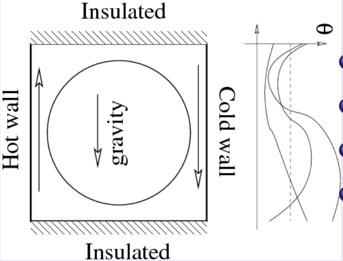
Uncertain boundary conditions



- $\boldsymbol{u} = 0$ on Γ.
- $\partial\theta(x, y = 0, 1)/\partial y = 0$.
- $\theta(x = 0, y) = 1/2$.

Governing equations

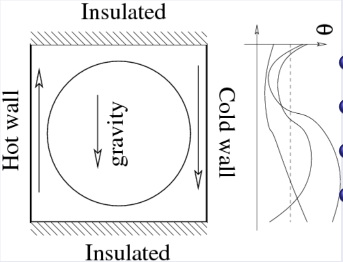
Uncertain boundary conditions



- $\boldsymbol{u} = 0$ on Γ.
- $\partial\theta(x, y = 0, 1)/\partial y = 0$.
- $\theta(x = 0, y) = 1/2$.
- $\theta(x = 1, y, \omega) = -1/2 + \theta'(y, \omega)$.

Governing equations

Uncertain boundary conditions



- $\boldsymbol{u} = 0$ on Γ.
- $\partial\theta(x, y = 0, 1)/\partial y = 0$.
- $\theta(x = 0, y) = 1/2$.
- $\theta(x = 1, y, \omega) = -1/2 + \theta'(y, \omega)$.

$$\langle\theta'(y)\theta'(y')\rangle = \sigma_\theta^2 \exp[-|y - y'|/L], \quad \theta' \sim N(0, \sigma_\theta^2).$$

BC and solution representations

$$\theta'(y,\boldsymbol{\xi}) = \sum_{i=1}^{\mathrm{N}} \sqrt{\lambda_i}\widetilde{\theta}_i(y)\xi_i = \sum_{k=0}^{\mathrm{P}} \theta_k(y)\Psi_k(\boldsymbol{\xi}).$$

$$(\boldsymbol{u}, p, \theta)(\boldsymbol{\xi}) = \sum_{k=0}^{\mathrm{P}} (\boldsymbol{u}, p, \theta)_k \Psi_k(\boldsymbol{\xi}).$$

- $\xi_i \sim N(0,1) \longrightarrow$ Hermite polynomials.
- Stochastic dimension N.
- Expansion order $\mathrm{No} \longrightarrow \mathrm{P} + 1 = (\mathrm{N} + \mathrm{No})!/(\mathrm{N}!\mathrm{No}!)$.

Galerkin projection

Implementation and solver

BC and solution representations

Galerkin projection

$$\frac{\partial \boldsymbol{u}_i}{\partial t} + \sum_{j=0}^{\mathrm{P}} \sum_{k=0}^{\mathrm{P}} \boldsymbol{u}_j \cdot \nabla \boldsymbol{u}_k \frac{\langle \Psi_i \Psi_j \Psi_k \rangle}{\langle \Psi_i \Psi_i \rangle} = -\nabla p_i + \frac{\mathrm{Pr}}{\sqrt{\mathrm{Ra}}} \nabla^2 \boldsymbol{u}_i + \mathrm{Pr}\,\theta_i \boldsymbol{y}$$

$$\frac{\partial \theta_i}{\partial t} + \sum_{j=0}^{\mathrm{P}} \sum_{k=0}^{\mathrm{P}} \boldsymbol{u}_j \cdot \nabla \theta_k \frac{\langle \Psi_i \Psi_j \Psi_k \rangle}{\langle \Psi_i \Psi_i \rangle} = \frac{1}{\sqrt{\mathrm{Ra}}} \nabla^2 \theta_i$$

$$\nabla \cdot \boldsymbol{u}_i = 0$$

- $\mathrm{P}+1$ coupled momentum and energy equations.
- $\mathrm{P}+1$ uncoupled divergence constraints and BCs.

Implementation and solver

BC and solution representations

Galerkin projection

Implementation and solver

Discretization

- Uniform grid, staggered arrangement and 2nd order FD.
- Semi-explicit second order Adams-Bashford time-scheme.

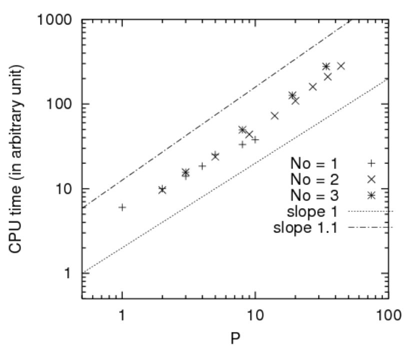
Incompressibility Treatment

- Prediction / Projection method (Chorin).
- FFT based solver for the elliptic pressure equations.

CPU : essentially projection of **uncoupled** modes :

Stochastic $\simeq (\mathrm{P}+1) \times$ deterministic.

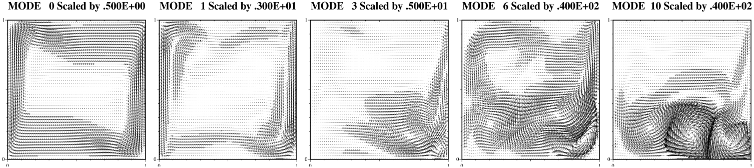
Convergence and performance (unsteady solver)



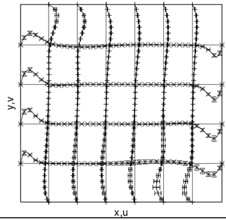
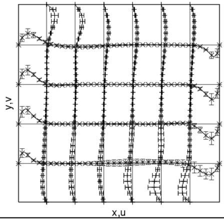
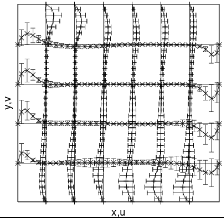
- $\mathrm{N} = 4 \sim 6$ is enough for $L \geq 1/3$.
- $\mathrm{No} = 3 \rightarrow$ relative error on variance $< 10^{-4}$.
- ~ 1000 times more efficient than MC (LHS).
- ~ 10 times more efficient than NISP + GH quadrature.

Le Maître *et al*, JCP (2001).

Example of velocity modes $\mathrm{Ra} = 10^6, L = 1 \ - \ \sigma_\theta = 0.25.$

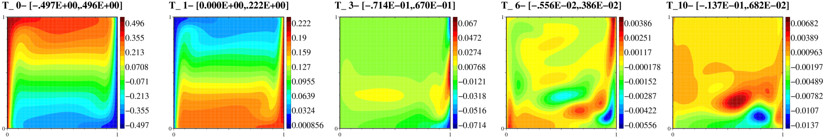


Uncertainty bars $L = 1.$

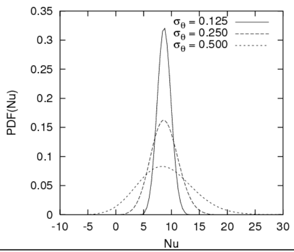
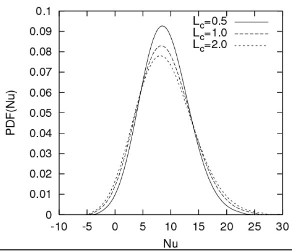
$\sigma_\theta = 0.125$	$\sigma_\theta = 0.25$	$\sigma_\theta = 0.5$
		

Le Maître *et al*, JCP (2002).

Example of temperature modes $\mathrm{Ra} = 10^6, L = 1 \; - \; \sigma_\theta = 0.25.$



Heat-transfert density

$L = 1 \; - \; \neq \sigma_\theta$	$\neq L \; - \; \sigma_\theta = 0.25$
	

Le Maître *et al*, JCP (2002).

Return

Natural convection — Low-Mach approximation

- Formulation (Najm *et al*, J. Comp. Phys., 1998 & 1999).

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= \frac{1}{\gamma T}\frac{dP}{dt} + \frac{1}{T}\left(\rho \boldsymbol{u}\cdot\nabla T - \frac{1}{\mathrm{Pr}\sqrt{\mathrm{Ra}}}\nabla\cdot(\kappa\nabla T)\right)\\
\frac{dP}{dt} &= -\gamma\int_\Omega \frac{1}{T}\left(\rho \boldsymbol{u}\cdot\nabla T - \frac{1}{\mathrm{Pr}\sqrt{\mathrm{Ra}}}\nabla\cdot(\kappa\nabla T)\right) d\Omega \Big/ \int_\Omega \frac{1}{T} d\Omega\\
\frac{\partial \rho u}{\partial t} &= -\frac{\partial \rho u^2}{\partial x} - \frac{\partial \rho u v}{\partial y} - \frac{\partial \Pi}{\partial x} + \frac{1}{\sqrt{\mathrm{Ra}}}\Phi_x\\
\frac{\partial \rho v}{\partial t} &= -\frac{\partial \rho u v}{\partial x} - \frac{\partial \rho v^2}{\partial y} - \frac{\partial \Pi}{\partial y} + \frac{1}{\sqrt{\mathrm{Ra}}}\Phi_y - \frac{1}{\mathrm{Pr}}\frac{\rho - 1}{2\epsilon}\\
T &= \frac{P}{\rho}
\end{aligned}
$$

- Difficulty : **non-linearities**

❒ **Exact inversion of the Galerkin product**.

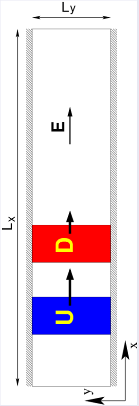
❒ **Exact mass-conservation** (mean sense is not enough).

Le Maître *et al.*, J. Sci. Comp., 2004.

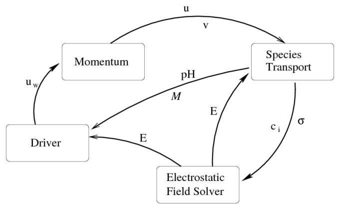
Return

Electrophoresis — Debusschere *et al*, Phys. Fluids (2003)

Problem



Code structure



Multi-physics : NS, diffusion convection, electro-osmotic flow, chemistry (finite & infinite rates).

Uncertainties

- ✔ ζ potential (BCs).
- ✔ Tension at channel ends.
- ✔ Reaction rates.
- ✔ Initial conditions.

Spectral UQ (Galerkin)

Respective influences of $\neq$ uncertainty sources.

Stochastic spectral methods for uncertainty quantification

Methodological developments

- 90s : Wiener-Hermite expansion of model solutions (Ghanem & Spanos).
- Applications to linear models (elasticity, thermal sciences, porous media, ...)
- 2000 : application to non-linear models : Navier-Stokes equations, porous media, reacting flows.
- 2004 : development of alternative expansion basis (generalised polynomial chaos, piecewise polynomial expansions, wavelets).
- Essentially rely on Eulerian formulations/models.

Are spectral expansions amenable to Lagrangian models ?

Lagrangian techniques for Navier-Stokes

Particle methods

- Solve (incompressible) N-S equations in rotational form.
- Theoretically well grounded.
- Deal with complex/moving boundary problems, infinite domains, . . .
- Immediate extension to low diffusivity/inviscid flows without requiring stabilisation or flux limiters.
- Handle transport and reactions.

Can we extend particle methods to propagate uncertainty ?

▸ Zap determ

2D incompressible Navier-Stokes equations

Rotational Form

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + \nabla \cdot (\boldsymbol{u}\omega) = \nu \Delta \omega, \\ \Delta \psi = -\omega, \\ \boldsymbol{u} = \nabla \wedge (\psi \boldsymbol{e}_z), \\ \omega(\boldsymbol{x}, 0) = (\nabla \wedge \boldsymbol{u}(\boldsymbol{x}, 0)) \cdot \boldsymbol{e}_z \\ \boldsymbol{u}, \omega \to 0 \quad \text{as } |\boldsymbol{x}| \to \infty. \end{cases}$$

Velocity kernel (Biot-Savart)

$$\boldsymbol{u} = \frac{-1}{2\pi} \mathcal{K} \star \omega = \frac{-1}{2\pi} \int_{\mathbb{R}^2} \mathcal{K}(\boldsymbol{x}, \boldsymbol{y}) \wedge (\omega \boldsymbol{e}_z) d\boldsymbol{y}, \quad \mathcal{K}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x} - \boldsymbol{y}) / |\boldsymbol{x} - \boldsymbol{y}|^2.$$

Particle approximation

Smooth approximation

Particles : position $\boldsymbol{X}_i(t)$, circulation $\Gamma_i(t)$, core size ϵ :

$$\omega(\boldsymbol{x}, t) = \sum_{i=1}^{\mathrm{Np}} \Gamma_i(t) \zeta_\epsilon(\boldsymbol{x} - \boldsymbol{X}_i(t)), \quad \lim_{\epsilon \longrightarrow 0} \zeta_\epsilon(\boldsymbol{x}) = \delta(\boldsymbol{x}).$$

Solution technique

Split convection and diffusion processes :

- Convection : transport particles with flow velocity.
- Diffusion : update particle circulations to account for diffusion (Particle Strength Exchange method).

▸ Zap details

Solution method

Convection step

$$\frac{d\boldsymbol{X}_i}{dt} = \frac{-1}{2\pi} \sum_{j=1}^{\mathrm{Np}} \Gamma_j \mathcal{K}_\epsilon(\boldsymbol{X}_i, \boldsymbol{X}_j), \qquad \frac{d\Gamma_i}{dt} = 0.$$

- $\mathcal{K}_\epsilon$: regularised Biot-Savart kernel.
- Reduce to ODE, but complexity in $\mathcal{O}(\mathrm{Np}^2)$.

Acceleration of velocity computation

- Multipoles expansion $\rightarrow \mathcal{O}(\mathrm{Np})$.
- **Particle-mesh techniques** :
  1. Project circulations Γ_i on an Eulerian mesh.
  2. Solve $\nabla^2 \Psi = -\omega$ (using FFT based solver for instance).
  3. Interpolate at $\boldsymbol{X}_i$ to obtain particle velocities.

Solution method

Integral representation of differential operators

Let $\eta(\boldsymbol{x})$ a radial function such that

$$\int_{\mathbb{R}^2} x^2\eta(\boldsymbol{x})d\boldsymbol{x} = \int_{\mathbb{R}^2} y^2\eta(\boldsymbol{x}) = 2,$$

$$\int_{\mathbb{R}^2} x^{\alpha_1}y^{\alpha_2}\eta(\boldsymbol{x})d\boldsymbol{x} = 0, \quad 1 \leq \alpha_1 + \alpha_2 \leq m+1, \ \alpha_1, \alpha_2 \neq 2,$$

then for positive integer multi-index β and $\eta_\epsilon(\boldsymbol{x}) \equiv \eta(\boldsymbol{x}/\epsilon)/\epsilon^2$ we have

$$\frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\ldots\partial x_d^{\beta_d}}f(\boldsymbol{x}) = \frac{1}{\epsilon^{|\beta|}}\int [f(\boldsymbol{y})+(-1)^{|\beta|+1}f(\boldsymbol{x})]\eta_\epsilon^{(\beta)}(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{y}+\mathcal{O}(\epsilon^m).$$

Degond & Mas-Gallic (1989), Eldredge *et al* (2002).

Solution method

Diffusion term

$$\frac{d\Gamma_i}{dt} = \nu \sum_{j=1}^{\mathrm{Np}} \mathcal{L}(\boldsymbol{X}_i - \boldsymbol{X}_j) S \left[\Gamma_j - \Gamma_i\right].$$

- Use compact functions η so only particles within a few core-size distances contribute.

Summary

$$\frac{d\boldsymbol{X}_i}{dt} = \frac{-1}{2\pi} \sum_{j=1}^{\mathrm{Np}} \Gamma_j \mathcal{K}_\epsilon(\boldsymbol{X}_i, \boldsymbol{X}_j),$$

$$\frac{d\Gamma_i}{dt} = \nu \sum_{j=1}^{\mathrm{Np}} \mathcal{L}(\boldsymbol{X}_i - \boldsymbol{X}_j) S \left[\Gamma_j - \Gamma_i\right].$$

Direct spectral expansion : the bad way !

Set **both particle positions and circulations as uncertain** :

$$\boldsymbol{X}_i(t,\xi) = \sum_k [\boldsymbol{X}_i]_k(t)\Psi_k(\xi), \quad \Gamma_i(t,\xi) = \sum_k [\Gamma_i]_k(t)\Psi_k(\xi).$$

Apply Galerkin projection to particle problem :

$$\left\langle \Psi_k^2 \right\rangle \frac{d[\boldsymbol{X}_i]_k}{dt} = \frac{-1}{2\pi} \sum_{j=1}^{\mathrm{Np}} \left\langle \Psi_k(\xi)\Gamma_j(\xi)\mathcal{K}_\epsilon(\boldsymbol{X}_i(\xi), \boldsymbol{X}_j(\xi)) \right\rangle ,$$

$$\left\langle \Psi_k^2 \right\rangle \frac{d[\Gamma_i]_k}{dt} = \left\langle \Psi_k(\xi)\nu(\xi) \sum_{j=1}^{\mathrm{Np}} \mathcal{L}(\boldsymbol{X}_i(\xi) - \boldsymbol{X}_j(\xi)) S\left[\Gamma_j(\xi) - \Gamma_i(\xi)\right] \right\rangle .$$

- Requires stochastic projection of the kernels.
- Fast algorithms for velocity estimation are impossible.

Untractable problem

Continuous stochastic problem : a better approach

Let's go back to the continuous vorticity equation :

$$\frac{\partial \omega(\xi)}{\partial t} + \boldsymbol{u}(\xi)\boldsymbol{\nabla}\omega(\xi) = \nu(\xi)\nabla^2\omega(\xi), \quad \omega(\boldsymbol{x},t,\xi) = \sum_k [\omega]_k(\boldsymbol{x},t)\Psi_k(\xi).$$

The Galerkin projection gives :

$$\frac{\partial [\omega]_k}{\partial t} + \sum_{i,j} C_{ijk}[\boldsymbol{u}]_i \boldsymbol{\nabla}[\omega]_j = \sum_{i,j} C_{ijk}[\nu]_i \nabla^2[\omega]_j, \quad C_{ijk} = \frac{\langle \Psi_i \Psi_j \Psi_k \rangle}{\langle \Psi_k^2 \rangle},$$

or, since by convention $\Psi_0 = 1 \Rightarrow C_{0jk} = \delta_{jk}$ and

$$\frac{\partial [\omega]_k}{\partial t} + [\boldsymbol{u}]_0 \boldsymbol{\nabla}[\omega]_k = -\sum_{i\neq 0,j} C_{ijk}[\boldsymbol{u}]_i \boldsymbol{\nabla}[\omega]_j + \sum_{i,j} C_{ijk}[\nu]_i \nabla^2[\omega]_j.$$

- Stochastic modes are **convected with the mean flow** $[\boldsymbol{u}]_0$.
- Interactions with other modes are treated as **source terms using integral approximations** (PSE).

Particle approximation of the stochastic problem

Particles with **stochastic strengths** $\Gamma_i(t,\xi) = \sum_k [\Gamma_i]_k(t)\Psi_k(\xi)$.

$$
\begin{aligned}
\frac{d\boldsymbol{X}_i}{dt} &= [\boldsymbol{U}_i]_0, \\
\frac{d[\Gamma_i]_k}{dt} &= -\sum_{j=1}^{\mathrm{Np}}\sum_{l=1}^{\mathrm{P}}\sum_{m=0}^{\mathrm{P}} C_{klm} S\left\{\mathcal{G}^x(\boldsymbol{X}_i-\boldsymbol{X}_j)\left([U_i]_l[\Gamma_i]_m+[U_j]_l[\Gamma_j]_m\right)\right. \\
&+ \left.\mathcal{G}^y(\boldsymbol{X}_i-\boldsymbol{X}_j)\left([V_i]_l[\Gamma_i]_m+[V_j]_l[\Gamma_j]_m\right)\right\} \\
&+ \sum_{j=1}^{\mathrm{Np}}\sum_{l=0}^{\mathrm{P}}\sum_{m=0}^{\mathrm{P}} C_{klm} S[\nu]_l \mathcal{L}(\boldsymbol{X}_i-\boldsymbol{X}_j)\left([\Gamma_j]_m-[\Gamma_i]_m\right), \\
[\boldsymbol{U}_i]_k &= \frac{-1}{2\pi}\sum_{j=1}^{\mathrm{Np}}[\Gamma_j]_k \mathcal{K}_\epsilon(\boldsymbol{X}_i,\boldsymbol{X}_j).
\end{aligned}
$$

- Kernels are evaluated only once for all modes.
- Fast algorithms for velocity computation are still possible.
- Formulation is conservative.

Lagrangian formulation Le Maître and Knio, J. Comp. Phys. (2007)

Particle method

Particles with

- deterministic positions,
- stochastic strengths (circulation & heat).

Time-integration : RK-3

- Particles convected by the mean flow.
- Integral representation of stochastic modes interactions.

Code efficiency

- Stable and diffusion free convection step.
- Fast algorithms for stochastic velocity calculation (*e.g.* FFT based, multipole expansion) : $\mathcal{O}(n \log n)$.
- Conservative method (regridding).

Results (I) — Convection of a passive scalar

Stochastic equations

$$\frac{\partial c}{\partial t} + \boldsymbol{U} \cdot \nabla c = 0,$$

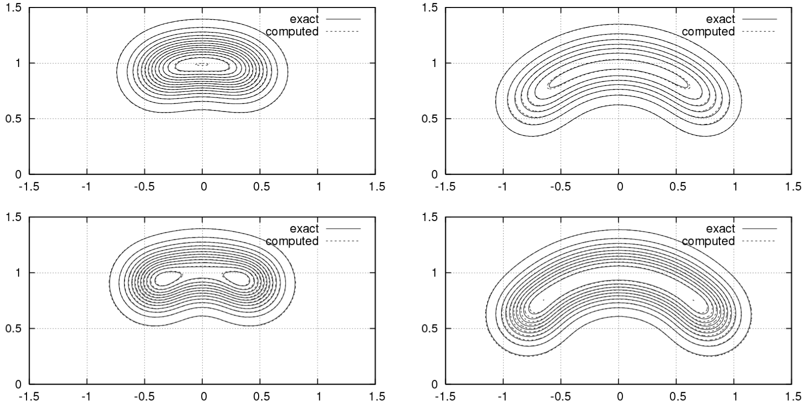
$$c(\boldsymbol{x}, t, \xi) = \exp\left[-\|\boldsymbol{x} - \boldsymbol{x}_0\|^2 / \pi d^2 \|\boldsymbol{x}_0\|\right], \quad \boldsymbol{x}_0 = \boldsymbol{e}_y,$$

$$\boldsymbol{U}(\boldsymbol{x}, \xi) = -(1 + 0.075\xi)\boldsymbol{x} \wedge \boldsymbol{e}_z, \quad \xi \sim U[-1, 1].$$

Discretization

- Particle positions $\boldsymbol{X}_i(t)$, $\epsilon = 0.025$.
- Particle strengths $C_i(t, \xi) = \sum_k [C_i]_k(t)\Psi_k(\xi)$.
- Stochastic basis : Legendre polynomial.
- Stochastic order up to $\mathrm{No} = 20$.
- RK-3 with $\Delta t = 2\pi/400$.

Mean and Standard deviation of $c(\boldsymbol{x}, t, \xi)$.



Mean (top row) and standard deviation (bottom row) of the scalar field after 1 revolution (left) and 2 revolutions (right). $\mathrm{No} = 20$.

Results (II)

Evolution of a radial vortex

Equations

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega = \nu \nabla^2 \omega,$$

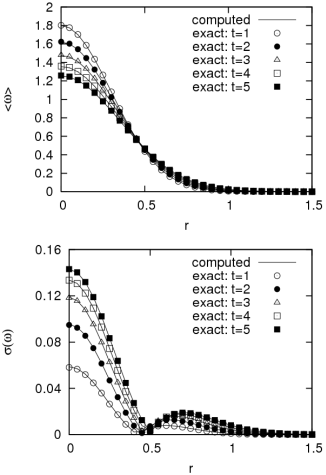
$$\omega(\boldsymbol{x}, t = 0) = \frac{\exp[-\|\boldsymbol{x}\|^2/d]}{\pi d},$$

$$\nu = 0.005 + 0.0025\xi, \quad \xi \sim U(-1, 1).$$

Discretization

- $\epsilon = 0.05$, remeshing every 10 iterations.
- Simulation for $t \in [0, 30]$, $\Delta t = 0.02$ with RK-3.
- Velocities computed with particle-mesh scheme $h_g = \epsilon$.
- Wiener Legendre expansion with $\mathrm{No} = 5$.
- Check the invariants of the flow.

Mean and Standard deviation of $\omega(\boldsymbol{x}, t, \xi)$.



Mean (top row) and standard deviation (bottom row) at different times.

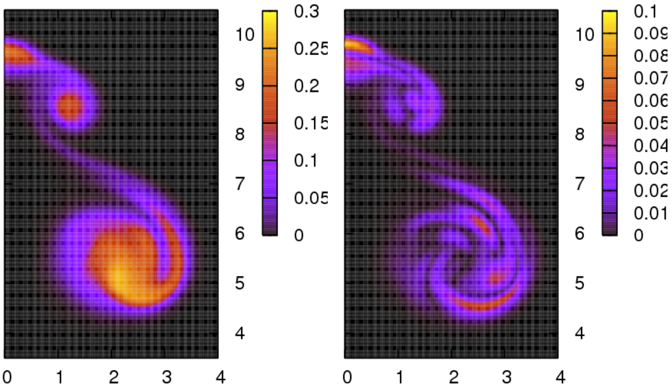
Equations

- Evolution of a compact hot patch of air in infinite medium.
- Boussinesq approximation : incompressible Navier-Stokes $+$ buoyancy terms and heat transport equation.
- Uncertainty and the Rayleigh number in the $\mathrm{Ra} \sim U[2.10^5, 3.10^5]$.

Discretization

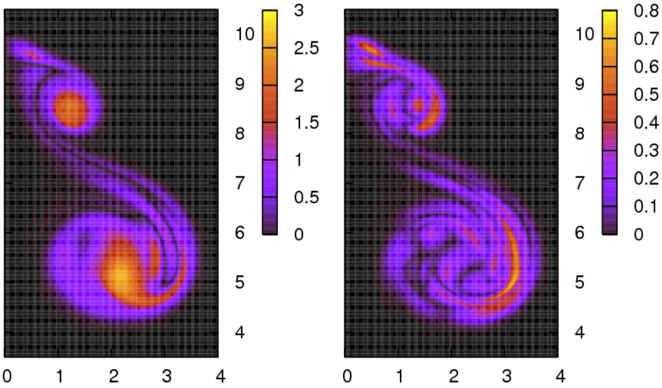
- $\epsilon = 1/30$.
- Simulation for $t \in [0, 28]$, $\Delta t = 0.2$ with RK-2.
- Remeshing every 4 iterations : $\mathrm{Np} > 200,000$ at the end of the simulation.
- Velocities computed with particle-mesh scheme $h_g = \epsilon$.
- Wiener Legendre expansion with up to $\mathrm{No} = 12$.

Mean and Standard deviation of the temperature field.



Temperature mean (left) and standard deviation (right)at $t = 20$.

Mean and Standard deviation of the vorticity field.



Vorticity mean (left) and standard deviation (right)at $t = 20$.

Return

(Non-intrusive techniques) Regression

- Let $\{\xi^{(1)}, \dots, \xi^{(m)}\}$ be the set of regression points, such that $\xi^{(i)} \in \Omega_\xi$, $i = 1, \dots, m$.
- Let $S^{(i)}$ be the solution of deterministic problem
$$\mathcal{M}\left(s^{(i)}, D(\xi^{(i)})\right) = 0, \text{ for } i = 1, \dots, m.$$
- Determine S_k, $k = 0, \dots, \mathrm{P}$, that minimizes the distance
$$d^2 = \sum_{i=1}^{m} w_i \left(S^{(i)} - \sum_{k=0}^{\mathrm{P}} S_k \Psi_k\left(\xi^{(i)}\right)\right)^2.$$

Advantages/issues

Return

- Works with a subset of the solution or by-products.
- Convergence with number of regression points m.
- Selection of the regression points.
- Error estimate.

Non intrusive projection

Make use of the orthogonality of the basis :

$$\langle S\Psi_k \rangle = \left\langle \Psi_k^2 \right\rangle S_k = \int_{\Omega_\xi} S(\xi)\Psi_k(\xi)p(\xi)d\xi.$$

Computation of $\mathrm{P}+1$ N-dimensional integrals

Non intrusive projection

Make use of the orthogonality of the basis :

$$\langle S\Psi_k \rangle = \left\langle \Psi_k^2 \right\rangle S_k = \int_{\Omega_\xi} S(\xi)\Psi_k(\xi)p(\xi)d\xi.$$

Computation of $\mathrm{P}+1$ N-dimensional integrals

(Quasi) Monte-Carlo sampling

$$\langle S\Psi_k \rangle \approx \frac{1}{m}\sum_{i=1}^{m} w^{(i)} S\left(\xi^{(i)}\right)\Psi_k\left(\xi^{(i)}\right).$$

- Convergence rate.
- Error estimate
- Optimal sampling strategy.

Non intrusive projection

Make use of the orthogonality of the basis :

$$\langle S\Psi_k \rangle = \left\langle \Psi_k^2 \right\rangle S_k \approx \sum_{i=1}^{N_Q} w^{(i)} S\left(\xi^{(i)}\right) \Psi_k\left(\xi^{(i)}\right).$$

Computation of $\mathrm{P}+1$ N-dimensional integrals

Numerical quadrature

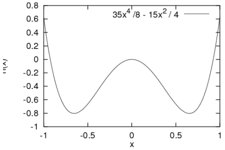
Quadrature points $\xi^{(i)}$ and weights $w^{(i)}$ obtained by

- full tensorisation of n points 1-D quadrature formula (*e.g.* Gauss formula) : $\boxed{N_Q = n^{\mathrm{N}}}$.
- partial tensorization of nested 1-D quadrature formula (Féjer, Clenshaw-Curtis) : $\boxed{N_Q < n^{\mathrm{N}}}$.
- Cost for large stochastic dimension N.
- Projection of non-polynomial solutions.

◂ Return

Example of GPC failure — Rolling-ball problem

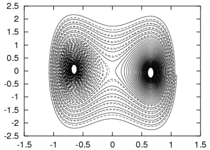
$$\frac{d^2X}{dt^2} + f\frac{dX}{dt} = -\frac{dh}{dX} \equiv -\frac{35}{2}X^3 + \frac{15}{2}X,$$



with friction $f \geq 0$.

Uncertain initial conditions :

$$X(t=0,\xi) = X_0 + \Delta X\xi, \quad \left.\frac{dX}{dt}\right|_{t=0} = 0,$$



with ξ U.D. on $[-1,1]$ (Legendre basis).

Solution : The system has two stable fixed points ($X^2 = 15/35$). Uncertainty in IC can lead to one fixed point **or** the other !

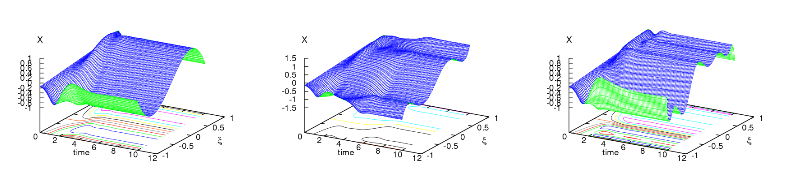
Stochastic solution may exhibit discontinuities.

Legendre solution — Rolling-ball problem

Parameters and solution method

$f = 2.$, $X_0 = 0.05$, $\Delta X = 0.2$; equation is time integrated using RK(3) and Galerkin projection.

Results for $\mathrm{No} = 3, 5$ and 9



Conclusion

Global polynomials (C^∞) can hardly represent discontinuous solution (Gibbs' oscillations).

Return

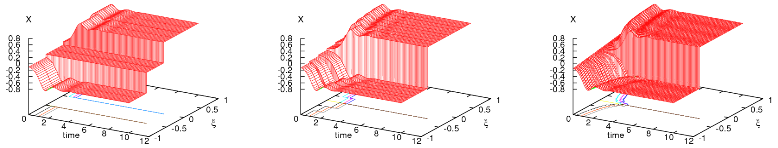
Wiener-Haar solution

Rolling-ball problem

Parameters and solution method

$f = 2.$, $X_0 = 0.05$, $\Delta X = 0.2$; equation is time integrated using RK(3) and Galerkin projection.

Results for $\mathrm{Nr} = 2, 3$ and 5



Remark

Details are not necessary evrywhere : adaptive method.

Return